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ELEMENTS
OF
TRIGONOMETRY,
PLANE and SPHERICAL:

WITH

The PRINCIPLES of PERSPECTIVE,

AND

PROJECTION of the SPHERE.

By JOHN WRIGHT.

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To the RIGHT HONOURABLE,

JOHN,

Earl of BREADALBANE and HOLLAND,
Viſcount of TAY and PAINTLAND,
Lord GLENORCHY, BENDERLOCH,
ORMLIE, and WICK, &c.

As a teſtimony of the zeal and prudence with which his Lordſhip has always ſtudied to promote the intereſt of his King and Country, both in the higheſt offices, and in a private capacity, and of his eſteem for whatever advances ſcience and virtue,

This treatiſe is inſcribed by,

His Lordſhip's

Moſt obedient

Humble ſervant,

JOHN WRIGHT.

PREFACE.

MY reaſons for publiſhing this treatiſe were not, that I had any diſcovery of my own to communicate in any of the ſubjects here treated, but becauſe I thought it proper to collect into one ſmall volume ſeveral things that relate directly or indirectly to trigonometry. I know of very little that can properly be called a diſcovery in the theory of trigonometry ſince Purbachius and Regiomontanus; the firſt of whom died in 1461, and the laſt in 1476: and the chief improvement in it ſince thoſe times is the application of the logarithms.

The trigonometry of the ancients, as appears from Ptolemy, was in form very different from ours. The trigonometrical canon of this author is conſtructed, by ſuppoſing the diameter of the circle to be divided into 120 equal parts, and by finding, in parts of the diameter, the chord of each degree and 60th part of a degree of the whole ſemicircle, or 180 degrees: each of the 120 parts of the diameter is ſuppoſed to be divided into 60 equal parts; and each of theſe again into 60 parts more, &c. See, from p. 8. to p. 17. of the great conſtruction, edit. 1538. At what time the ancient trigonometry came to be reduced to the regular form in which we find it in Ptolemy, I have not hitherto been able to diſcover; but it ſeems to have been poſterior to Ariſtarchus the Samnite, who flouriſhed about 280 years before the Chriſtian æra. One would think, that if there had been a trigonometrical canon then eſtabliſhed, it would have been uſed by this author in his treatiſe of the magnitude and diſtance of the ſun and moon; but no ſuch thing appears. For, in the 7th propoſition of that treatiſe, the way in which he expreſſes the angle contained by ſtraight lines drawn from the centres of the earth and moon to that of the ſun, when the moon is in the quadrature, is, that it is the 30th part of a right angle. It would appear alſo, that this invention was poſterior to the time of Archimedes: for, in his treatiſe called the *Arenarius*, the way in which he expreſſes the angle ſubtended by the diameter of the ſun is, that it is leſs than the 164th but greater than the 200dth part of a right angle. Probably the invention was by Hipparchus, who began to flouriſh about 50 years after the death of Archimedes; that is, about 160 years before the Chriſtian æra: for we are told, that Hipparchus wrote a treatiſe upon the uſe of chords. The Arabians afterwards, though it is uncertain at what time, altered the form of the ancient trigonometry. The alteration appears to have been made before the time of Albatenius, who flouriſhed about the latter end of the ninth century. They made uſe of the radius of the circle inſtead of the diameter; but continued to divide it into 60th parts as before: they made uſe of half the chord, which is now called the ſine, inſtead of the chord itſelf, and found it in parts of the radius; and they reduced the caſes of triangles to ſimple propoſitions of four proportionals. The reaſon of the name *ſinus*, ſine, is ſaid to be, that the halves of the chords, *ſemiſſes inſcriptarum*, might often be contracted thus, *S. Inſ.*; and that the ignorant copiators made one word, *ſinus*, of both. M. *Montucla*, in his *Hiſtoire de Mathématiques*, ſays, that he had this etymology from M. Godin.

Purbachius ſaw it would be more convenient to have the radius divided into decimal parts than into ſexageſimals: he made the radius 6000000, and calculated the ſines in decimal parts of it, inſtead of the ſixtieth parts into which the calculations are made by Ptolemy and the Arabians. Regiomontanus, who was the ſcholar of Purbachius, made this farther alteration in the calculation of the ſines, that he ſuppoſed the radius to be unit, and to be divided into decimal parts. The radius of Regiomontanus therefore is 10000000. The ſame eminent perſon alſo added to trigonometry the uſe of tangents, and calculated tables in parts of the radius for every degree and minute of the quadrant, and diſcovered propoſitions, by means of which the two laſt caſes of oblique-angled ſpherical triangles are ſolved. Rheticus, ſoon after Regiomontanus, added the uſe of ſecants to trigonometry.

Vieta, who flouriſhed towards the latter end of the ſixteenth century, improved trigonometry, as he did the other parts of geometry and mathematics, by his elegant manner of conceiving and demonſtrating propoſitions. He is likewiſe the firſt author in whom I found right-angled ſpherical triangles reduced to plane triangles in order to ſolution; a method which renders the doctrine of ſpherical triangles much more intelligible to learners. I have followed his method in that particular in this treatiſe, and ſupplied the caſes that are not taken notice of by him.

Every body knows how much the diſcoveries of Baron Napier contributed to the perfection of trigonometry. By the logarithms, the calculations of plane and ſpherical trigonometry are performed with the greateſt eaſe; and by the rules he invented, the numerous caſes of ſpherical trigonometry are eaſily retained in the memory. Since his time, the principal writers of trigonometry have had chiefly in view how to conſtruct tables of logarithms in the readieſt manner, and to adapt them to trigonometry.

The modern diſcoveries in mathematics, viz. algebra and fluxions, have likewiſe cauſed alterations in the method of conſtructing trigonometrical tables: for, by means of theſe, the tables can be conſtructed in a much ſhorter time. It is difficult, however, to make ſuch as are only learning trigonometry underſtand how trigonometrical tables are conſtructed by algebra and fluxions; and as plane trigonometry is capable of being ſufficiently underſtood, and commonly taught immediately after learning the firſt ſix books of Euclid, it is ſurely proper that a method of conſtructing tables ſhould be taught at the ſame time. Whatever method therefore the learned may take in order to conſtruct tables of logarithms, and of ſines and tangents, in the moſt expeditious manner, it is neceſſary that a method deduced from the firſt ſix books of Euclid, and common arithmetic, be explained to the learner who has advanced no farther: and as the logarithmic ſines, tangents, &c. are the logarithms of the natural ſines, tangents, &c. a learner can have but a very imperfect underſtanding of logarithmic ſines and tangents before he is acquainted with the natural.

As I believe ſome trigonometrical tables we have already to be very ſufficient, I thought it only neceſſary to deſcribe methods by which they may be conſtructed, without enumerating many, or examining which is preferable. What I have ſaid of the hyperbolical logarithms, is in order to give the learner a view of the origin and progreſs of logarithms; and I have diſtinguiſhed it from the other part of the ſection; becauſe, in order to underſtand it, the reader muſt be acquainted with the higher geometry.

The ſcales and the ſector are very uſeful inſtruments for ſolving the caſes of trigonometry, when no great accuracy is neceſſary; I have therefore deſcribed how they are conſtructed and uſed in trigonometry.

The ſections that treat of perſpective, and the projection of the ſphere, were intended to make ſpherical trigonometry more eaſily underſtood. I have obſerved, that beginners have commonly great difficulty to underſtand what is the meaning or uſe of a ſpherical triangle when they ſee the repreſentation of it upon a plane, without being taught how to project the different circles of a ſphere upon a plane: and becauſe projection of the ſphere is but a particular caſe of perſpective, I found it of great advantage to make the learner firſt acquainted with perſpective, next with the projection of the ſphere, and laſtly with ſpherical triangles. Perſpective is curious and uſeful for its own ſake; and therefore I have treated it at greater length than is barely neceſſary for underſtanding the projection of the ſphere.

The reader will ſee, that I had no intention by this treatiſe to ſuperſede the uſe of any ſtandard book that is taught as a part of a regular courſe of geometry; but rather ſuppoſed, that the learner is acquainted with ſeveral of them before he can benefit any thing by this. In the firſt part (except what is ſaid of the hyperbolical logarithms) I have ſuppoſed the learner acquainted only with the firſt ſix books of Euclid, common arithmetic, and the extraction of the ſquare root. I take it for granted, that every teacher of Euclid explains the principles upon which the rules for extracting the ſquare root are founded when he is teaching the ſecond book of Euclid, and that he explains what is called *the rule of three* when he is teaching the fifth and ſixth books. In the ſecond part, the learner is ſuppoſed to underſtand the eleventh and twelfth books of Euclid; eſpecially the firſt twenty propoſitions of the eleventh, and Theodoſius upon the ſphere, at leaſt the firſt book of it. In the ſecond part too, I have made ſome references to ſome of the conic-ſection figures; particularly to the ellipſis. Becauſe ſpheric geometry is, in the univerſities of Scotland at leaſt, taught in the ſame claſs with conic ſections, I did not think it neceſſary ſo much as to ſhew how an ellipſis is deſcribed *; becauſe two diſtinct parts of a courſe of geometry ſhould not in the leaſt interfere with one another.

In the ſections that treat of the conſtruction of natural ſines, tangents, and ſecants, I have nearly followed the methods of Regiomontanus and Baron Wolfius. But in order to find the ſine of 36°, I was under a neceſſity of inſerting four propoſitions; two of them from the thirteenth book of Euclid, and two from Ptolemy. Wolfius finds the ſine of 36° by means of his own geometry. What I have ſaid of the common logarithms is alſo according to him.

For what I have ſaid of the hyperbolical logarithms, I was much indebted to the valuable work of the very learned and judicious M. Montucla, intitled, *Hiſtoire des Mathématiques*, printed at Paris in 1758, in two volumes quarto. He treats that ſubject ſo diſtinctly, that I had little more to do, but to tranſlate what he has ſaid into Engliſh, and to ſupply ſome demonſtrations, which he had properly enough omitted, becauſe he was writing for proficients in geometry. In one of theſe demonſtrations, I was ſomewhat at a loſs, viz. the one in article 46.; where it is ſhewn, that the ſpaces contained by the hyperbola, the ordinates to the aſymptote, and the differences of the aſymptotes, are equal, when the aſymptotes are taken in continued geometrical proportion. I could not procure a ſight of Gregory of St Vincent's own works; neither could I meet with the demonſtration given by him in another author; I therefore gave the ſhorteſt and moſt natural demonſtration that occurred.

---

* In a Synopſis of practical mathematics, lately publiſhed at Edinburgh, I obſerved a problem propoſed, "To deſcribe an ellipſis." The ſolution that is given is, to join two equal arcs of equal leſſer circles to two equal arcs of two equal greater circles: the figure made up of theſe four arcs is ſaid to be an ellipſis. It is unneceſſary to ſhow, that ſuch a figure cannot be an ellipſis, and that no part of an ellipſis is circular, however great an affinity there is between the two curves.

The ſection that treats of the ſcales and ſector, is in a good meaſure according to Bion, and a ſmall treatiſe upon mathematical inſtruments by J. Robertſon, F. R. S.

The ſection that treats of perſpective is for the moſt part in the manner of Graveſend, in a ſmall treatiſe, intitled, *Eſſai de Perſpective*, printed in 1711; and of Wolfius, in the third volume of his *Opera Mathematica*. Theſe two authors, eſpecially the laſt, were the moſt conciſe and ſatisfactory I met with upon the ſubject, though I did not invariably adhere to either. In the end of the 25th article of perſpective, I have ſaid, that the perſpective of a circle in the geometrical plane is made up of two ſemi-ellipſes upon the ſame greater axis, but having unequal leſſer axes. I have not added the demonſtration of this: The reader may conſult the 28th and 29th of Serenus, *De ſect. cylin.* as ſubjoined by Dr Halley to his edition of Apollonius Pergæus's conic ſections.

In the ſections that treat of plane and ſpherical trigonometry in particular, and projection of the ſphere, I had no particular author to follow. In the right-angled plane and ſpherical triangles, I have given the enunciations of the propoſitions ſomething differently from what is commonly done. For inſtance, in the firſt propoſition of plane triangles, I have ſaid, If the hypothenuſe be made radius, each of the ſides ſhall be the ſine of the angle oppoſite to itſelf; inſtead of ſaying, The hypothenuſe is to a ſide, as radius to the ſine of the angle oppoſite to that ſide. I thought this manner of expreſſion more agreeable to the definitions:

tions: for which reaſon, the propoſitions of right-angled ſpherical triangles are enunciated to the ſame purpoſe; and I have ſet down in corollaries the manner of expreſſing the ratios of radius, ſine, tangent, in the propoſitions, to radius, ſine, tangent, in the tables, and upon the ſcales and ſector.

In the propoſitions by which the caſes of oblique-angled plane triangles are ſolved, viz. the 4th, 5th, and 6th, of the firſt part, I have made uſe of figures and demonſtrations different from any author I have ſeen. The propoſition in other authors correſponding to my 4th, requires two different equal radii. Beginners are ſometimes perplexed by this: I have therefore contrived the figure ſo, that one circle ſerves the purpoſe. The 5th alſo, which was of more difficult invention, I have contrived ſo, that one and the ſame circle ſerves, and that the tangent of the angle which is half the ſum of the two angles, is in the ſame ſtraight line with the tangent of the angle which is half their difference, and contiguous to it. The demonſtration of the 6th propoſition, which I thought more geometrical than that which is commonly given, I made up of part of the 120th propoſition of the ſeventh book of Pappus Alexandrinus, and of a corollary or inference that follows manifeſtly from the 5th propoſition of the ſecond book of Euclid. Beſides theſe three propoſitions, there are ſeveral other demonſtrations here and there, which are different from any I have ſeen in like caſes. I have deduced the demonſtrations of Napier's two rules as corollaries from the propoſitions upon which they depend, in order to ſave the learner's time. The laſt two propoſitions of ſpherical triangles are from Wolfius.

During the time that this treatiſe was in the preſs, ſeveral other treatiſes appeared, which, by their titles, ſeemed, in ſome meaſure, to interfere with it. But after examining two or three of them, I ſaw no reaſon to wiſh that any part of it had been ſuppreſſed, or the order altered. The deſign of them all ſeemed to be, to comprehend a ſort of univerſal courſe of mathematics in a ſmall ſize; and the propoſitions in them are without any demonſtrations. This treatiſe is almoſt confined to trigonometry, and intended for ſuch as ſtudy geometry regularly.

I always imputed the ſlowneſs with which young people comprehend natural philoſophy and aſtronomy, to their ignorance of geometry and trigonometry; and I thought the principles of perſpective might contribute to their underſtanding the plates of a book of natural philoſophy. Beſides, I obſerved, that many who underſtand a geometrical plan very well, have no idea at all of the geometry of viſibles.

The mathematical tables I had in view through the whole of this treatiſe were Sherwin's; becauſe any tables ſmaller than theſe cannot be of much uſe. For, beſides the great extent to which they are carried, they contain alſo the natural ſines, tangents, &c.; without which, the learner can have but a very confuſed idea of the tables of artificial ſines and tangents. In the caſes of trigonometry I put, I carried the calculations only the length of minutes. The reader, by means of Sherwin's tables at leaſt, can eaſily carry them the length of ſeconds, or thirds, if he pleaſes. In the projection of the ſphere, where no great accuracy is expected, I called the inclination of the ecliptic to the equator 23° and an half. If accuracy is required, the reader will conſult the lateſt obſervations for this inclination.

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ELEMENTS

OF

TRIGONOMETRY.

PLANE TRIGONOMETRY.

INTRODUCTION.

1. TRIGONOMETRY is an art founded upon mathematical truths, by which unknown ſides or angles of a triangle are found, by means of other ſides and angles that are given. A triangle, contained by three ſtraight lines, is called a *plane triangle*; and that part of trigonometry which is applied to it, is called *plane trigonometry*.

2. Right-angled plane trigonometry is that which is applied to right-angled plane triangles, as ABC, plate 1. fig. 1.; oblique-angled plane trigonometry is that which is applied to a plane triangle, none

of

of whoſe angles is a right angle, as ABC, fig. 2. & 3.

3. In a right-angled plane triangle ABC, fig. 1. the ſide AB ſubtending the right angle ACB, is called the *hypothenuſe*; and AC, CB, that contain the right angle, are called the *ſides*. Sometimes one of the ſides is called the *baſe*, and the other the *perpendicular*.

4. The circumference of a circle ACBD, fig. 4. is ſuppoſed to be divided into 360 equal parts, called *degrees*; each degree is ſuppoſed to be divided into 60 equal parts, called *minutes*; and each minute into 60 equal parts, called *ſeconds*, &c.

5. Any angle HEB at the centre of a circle, is ſaid to be an angle of as many degrees, minutes, and ſeconds, &c. as there are degrees, minutes, and ſeconds, in the circumference HB on which it ſtands. For example, if the circumference HB contain thirty degrees, or thirty of the equal parts whereof the whole circumference BCAD contains 360, then the angle HEB is an angle of thirty degrees. In like manner, the angle GEB is an angle of as many degrees as

there are degrees of the whole circumference BCAD contained in the circumference GHB, and the angle FEB is of as many degrees as there are degrees in the circumference FGHB, and ſo on. And becauſe [2. cor. 15. 1.] all the angles made by any number of ſtraight lines meeting in one point E, are, together, equal to four right angles, the whole circumference of a circle BCAD is the meaſure of four right angles. Hence the circumference BGC, or ninety degrees, viz. the fourth part of the whole circumference BCAD, is the meaſure of the right angle CEB; the ſemicircle BCA, an hundred and eighty degrees, is the meaſure of two right angles CEB, CEA, or of any two adjacent angles FEB, FEA [13. 1.].

DEFINITIONS.

6. Def. 1. Any ſtraight line HR, placed in a circle BCAD, fig. 4. is the chord of each of the circumferences or arcs HBR, HAR, cut off by the ſtraight line HR. In like manner, GS is the chord of each of the circumferences or

arcs GBS, GAS; FT, of FBT, FAT. A ſtraight line EB, drawn from the centre E, biſecting the chord HR, biſects alſo the arc HBR ſubtended by that chord. [4. 1. & 28. 3.]

7. Def. 2. The ſine of any arc HB, or of the angle HEB, meaſured by the arc HB, is HQ, viz. the half of the chord of double that arc. Likewiſe GP, FO, are, each of them, the ſines of each of the arcs GHB, FHB, or of the angles GEB, FEB, meaſured by theſe arcs.

The ſemidiameter or radius CE, (viz. the ſine of a quadrant or ninety degrees), is [15. 3.] the greateſt of all the ſines. FO, the ſine of any arc FHB leſs than a quadrant, is alſo the ſine of FCA; an arc as much greater than a quadrant as the other is leſs.

Any ſine FO, drawn from one extremity F of the arc FHB, or FCA, is [3. 3.] perpendicular to AB, the diameter that paſſes through the other extremity of the arc.

8. Def. 3. If K*a*, fig. 4. be drawn parallel to the chord HR, touching the circle in B, and meeting each of the ſtraight lines EH, ER, drawn from the

centre through the extremities of the arc HBR, the parts BK, B*a*, between the point of contact B, and the extremities K, *a*, are the tangents of the arcs or of the angles, whereof the half of the chord HR is the ſine. In the ſame manner, LB is the tangent of the arc GHB, or of the angle GEB; and MB is the tangent of the arc FHB, or of the angle FEB.

If the quadrant CB be biſected in G, BL, the tangent of its half GHB, (forty-five degrees), is equal to radius CE [34. 1.]

The tangent of a quadrant CGB is indefinite; for BM is [28. 1.] parallel to EC.

9. Def. 4. The ſtraight lines EK, E*a*, drawn from the centre to the extremities of the touching line K*a*, are the ſecants of the arcs or angles, whereof its parts, from the point of contact to the extremities, are the tangents. EL is the ſecant of GHB, and EM of FHB. The ſecant is always greater than radius; and the ſecant of a quadrant CGB being parallel to the tangent BM, is indefinite.

10. Def. 5. What any arc FHB wants of a quadrant, or any angle FEB of a right angle, is the complement of that

arc or angle. Thus, CF is the complement of the arc FHB, and FHB is the complement of the arc CF; the angle FEC is the complement of FEB, and FEB is the complement of FEC.

11. Def. 6. The coſine of any arc FHB, is the ſine of its complement, as FZ, which is equal to EO the diſtance of the ſine from the centre.

12. Def. 7. The cotangent of any arc FHB is the tangent of its complement, as CN.

13. Def. 8. The coſecant of any arc FHB is the ſecant of its complement, as EN.

14. Def. 9. The verſed ſine or ſagitta of any arc HB, is the exceſs of the radius above its coſine, as QB.

SECTION I.

Method of conſtructing tables of natural ſines, tangents, and ſecants.

15. THE radius of the circle is conſidered as unit, and is ſuppoſed to be divided into 10000000 or more decimal parts. The ſine, tangent, and

ſecant, of each degree and minute of the quadrant, is, by the following method, found in parts of the radius.

PROP. I. PROBLEM. *Fig.* 5.

16. *To find the ſine of* DB, *an arc of thirty degrees, in parts of the radius.*

Make BG equal to BD; join DG, and biſect it in H; DH or HG is the ſine of thirty degrees. Becauſe the ſide of a hexagon or chord of ſixty degrees is [cor. 15. 4.] equal to radius, and that the ſine of any arc is [def. 2. of this] half the chord of double the arc, therefore DH is the ſine of thirty degrees; and if the radius be 10000000, the ſine of thirty degrees ſhall be 5000000 of the ſame parts. Which was to be done.

PROP. II. PROB. *Fig.* 5.

17. *The ſine of any arc* DB *being given, to find its coſine* DF.

Let DH, the ſine of DB, meet the diameter AB in H; join CD; and becauſe the ſquare of CD is [47. 1.] equal to the ſquares of CH and HD, that is

[34. 1.],

[34. 1.], to the ſquares of DF and DH, and that the ſquares of CD, DH, are each of them given, (CD being radius); therefore the ſquare of DF, and the ſtraight line DF, are each of them given. For example, if the arc DB be thirty degrees, its ſine is [1. of this] 5000000 of the parts whereof radius is 10000000. If therefore the ſquare of 5000000 be ſubtracted from the ſquare of 10000000, the ſquare root of the remainder, viz. 8660254 ſhall be the number expreſſing the coſine DF, or the ſine of 60° in parts of the radius.

In the ſame manner, the coſine of any other arc is found in parts of the radius, if the ſine be given; or if the coſine be given, the ſine is found.

PROP. III. PROB. *Fig.* 5.

18. *The ſine of any arc* DB *being given, to find the ſine of its half.*

Join DB, which is therefore the chord of the arc DB; find [2. of this] CH the coſine of the arc DB, and ſubtract it from radius CB; the remainder is the verſed ſine HB. And becauſe the ſquares of HB and HD are [47. 1.] equal to the ſquare of DB, and that HB, HD, and their ſquares, are each of them given; therefore the ſquare of DB, and the ſtraight line DB, are each of them given. But the half of the chord DB is [def. 2. of this] the ſine of half the arc DB. Which was to be done.

Thus, if the arc DB be thirty degrees, its ſine DH being [1. of this] 5000000, and its coſine [2. of this] 8660254 parts of the radius, the ſine of its half, or of fifteen degrees, ſhall be found to be, in parts of the radius, 2588190.

PROP. IV. PROB. *Fig.* 5.

19. DH, *the ſine of any arc* DB, *being given, to find* DL, *the ſine of its double,* DBG.

Find CH [2. of this] the coſine of the arc DB; join CG. Becauſe DH, the ſine of the arc DB, is given, DG, the chord of double the arc, is [def.. 2. of this] given. And becauſe CHG and DLG are [def. 2. of this] each of them right angles, and that the angle CGD is common to the two triangles CGH, LGD, the triangles CGH, LGD, are [32. 1.] equiangular, and [4. 6.] the ſides containing the equal angles, proportionals; wherefore GC is to CH, as GD to DL: but the three firſt terms of the proportionals are each of them given; therefore the fourth term DL, that is the ſine of double the arc DB, is given. Which was to be done.

PROP. V. PROB. *Fig.* 4.

20. *To find* GP, *the ſine of* GHB, *an arc of forty-five degrees.*

Becauſe GP is the ſine of an arc of forty-five degrees, EP, which [34. 1.] is equal to the coſine of GHB, is alſo the ſine of forty-five degrees, and is therefore equal to GP; and becauſe the ſquare of EG is [47. 1.] equal to the ſquares of EP and PG, and that EP is equal to PG, the ſquare of EG is double of the ſquare of GP: but the ſquare of the radius EG is given; therefore the ſquare of GP, and the ſtraight line GP, are each of them given. The ſquare root of half the ſquare of the radius, or the ſine of forty-five degrees, in parts of the radius, is found to be 7071068. Which was to be done.

PROP.

PROP. VI. THEOR. *Fig.* 6.

Which is the 9th of the 13th book of Euclid.

21. *If* BD, *the ſide of a decagon, and* DE, *the ſide of a hexagon, inſcribed in the ſame circle* ABD, *be added together, the whole ſtraight line* BE, *made up of both, ſhall be divided in extreme and mean ratio in* D, *their common extremity.*

Find C, the centre of the circle, draw the diameter BA, and join CD, CE. Becauſe BD is the ſide of a decagon inſcribed in the circle ABD, the circumference AD is [28. 3.] quadruple of the circumference BD, and the angle ACD [33. 6.] quadruple of the angle BCD; but the angle ACD is [32. 1.] equal to the two angles CDB, CBD, and [5. 1.] double of one of them CDB; therefore the angle CDB is double of the angle DCB; and becauſe DE is [cor. 15. 4.] equal to DC, the angle DCE is [5. 1.] equal to the angle DEC, and the angle CDB [32. 1.], double of the angle DEC; wherefore the angle DEC is equal to the angle BCD, and the angle CBD is common to the two triangles CEB, DCB; therefore the angle ECB is [32. 1.] equal to the angle CDB; and [4. 6.] BE is to BC, as BC to BD: but [cor. 15. 4.] BC is equal to ED; therefore BE is to ED, as ED to DB. Q. E. D.

PROP. VII. THEOR. *Fig.* 7.

Which is the 10th of the 13th book of Euclid.

22. *The ſquare deſcribed upon* AB, *the ſide of a pentagon* ABCDE, *inſcribed in a circle, is equal to the ſquare deſcribed upon the ſide of a hexagon, together with the ſquare deſcribed upon* AH *or* HB, *the ſide of a decagon inſcribed in the ſame circle.*

Find the centre F, and join FB; FB is [cor. 15. 4.] equal to the ſide of a hexagon inſcribed in the circle: draw the diameter AFG; join FH; and draw FL at right angles to AH, meeting AB in M; and the circumference of the circle in K; and join HM.

Becauſe [28. 3.] the circumference ABC is equal to the circumference AED, and that ABCG, AEDG, are ſemicircles, the circumference CG is equal to the circumference GD, and CGD is double of CG:

but the circumference CGD is [28. 3.] equal to the circumference AHB, and the circumference AHB is double of the circumference AKH; therefore the circumference AKH is equal to the circumference CG. And becauſe FL is at right angles to AH, the angle AFK is [3. 3. & 8. 1.] equal to the angle KFH, and the circumference AK [26. 3.] to the circumference KH; wherefore the circumference GCB is double of the circumference BHK, and the angle GFB is [26. 3.] double of the angle BFK: but the angle GFB is [32. & 5. 1. or 20. 3.] double of the angle FAB; therefore the angles BFK, FAB, are equal: and the angle FBM is common to the two triangles ABF, MBF; therefore [32. 1.] they are equiangular; and [4. 6.] BA is to BF, as BF to BM; and the rectangle ABM is [17. 6.] equal to the ſquare of BF. Again, becauſe ML biſects HA at right angles, the angle LHM is [4. 1.] equal to the angle LAM; that is, (becauſe AH, HB, are equal, and 5. 1.) to the angle HBA; and the angle HAM is common to the two triangles HBA, HAM; therefore the triangles HBA, HAM, are equiangular; and [4. 6.] BA is to AH, as AH to

AM; and [17. 6.] the rectangle BAM is equal to the ſquare of AH: but the ſquare of BF was ſhown to be equal to the rectangle ABM; therefore the ſquares of BF and AH are equal to the rectangles ABM and BAM; that is, [2. 2.] to the ſquare of AB. Q. E. D.

PROP. VIII. THEOR. *Fig.* 8.

From Clau. Ptol. Μεγάλη Σύνταξις, p. 8. edit. 1538.

23. *If, from* D, *the centre of a circle* AGB, *a ſtraight line* DB *be drawn, at right angles to the diameter* AG, *meeting the circumference in* B, *and if the ſemidiameter* DG *be biſected in* E, *and from* EA, (*which* [7. 3.] *is greater than* EB) EC *be cut off equal to* EB, *the ſtraight line intercepted between* C *and* B *ſhall be equal to the ſide of an equilateral and equiangular pentagon inſcribed in the circle* AGB.

Becauſe GD is equally divided in E, and produced to C, the rectangle GCD, together with the ſquare of DE, is [6. 2.] equal to the ſquare of EC or EB; that is, [47. 1.] to the ſquares of DE and DB; wherefore the rectangle GCD, together with the ſquare of DE, is equal to the ſquares of DE and DB; take away the common ſquare of DE, and the rectangle GCD ſhall be equal to the ſquare of DB or DG; therefore [17. 6.] CG is to GD, as GD to DC, and CG is divided in extreme and mean ratio in D: but DG is [cor. 15. 4.] equal to the ſide of a hexagon inſcribed in the circle AGB; therefore [conv. 6. of this] DC is equal to the ſide of a decagon inſcribed in it. Again, becauſe the ſquare of CB is [47. 1.] equal to the ſquares of DB and DC, and that DB is equal to the ſide of a hexagon, and DC equal to the ſide of a decagon, CB is [conv. 7. of this] equal to the ſide of a pentagon inſcribed in the circle AGB. Q. E. D.

PROP. IX. PROB. *Fig.* 8.

From the ſame.

24. *To find, in parts of the radius,* CB *a ſtraight line equal to the ſide of a pentagon inſcribed in a circle* AGB.

Let the points B and C be found as in the preceding propoſition. And becauſe the radius DB or DG is given, its half DE is given; therefore the ſum of the ſquares of DB and DE, that is, [47. 1.] the ſquare of EB, is given; wherefore the ſtraight line EB or EC is given: and becauſe the ſtraight line EC, and a part of it DE, are each of them given, the remainder DC is given; wherefore the ſum of the ſquares of DC and DB, that is, [47. 1.] the ſquare of CB, and the ſtraight line CB, are each of them given. Which was to be done.

Cor. Hence the ſine of thirty-ſix degrees is found in parts of the radius. Becauſe the ſide of a pentagon inſcribed in a circle is [def. 1. of this] the chord of ſeventy-two degrees, its half is [def. 2. of this] the ſine of thirty-ſix degrees; which, in parts of the radius, is 5877283.

PROP. X. PROB. *Fig.* 9.

25. EF *and* AB, *the ſines of two given arcs* ED *and* AD, *being each of them given, to find the ſine of the half of the arc* EA, *which is the difference of the arcs* ED *and* AD.

Becauſe EF and AB, the ſines of the given arcs ED and AD, are each of them given, their coſines EH and AG are alſo [2. of this] each of them given. Let AG meet EF in L, and join AE.

Becauſe EF and AB, that is, LF [34. 1.], are each of them given, their difference EL is given; and becauſe AG and EH, that is, LG are each of them given, their difference AL is given; therefore the ſum of the ſquares of EL and LA, that is, [47. 1.] the ſquare of AE is given; wherefore the ſtraight line AE is given; but the half of the ſtraight line AE is [def. 2. of this] the ſine of half of the arc EA. Which was to be found.

PROP. XI. PROB. *Fig.* 10.

26. AB *and* EF, *the ſines of the given arcs* KB *and* KF, *whoſe difference does not exceed forty-five minutes, being each of them given, to find, nearly,* GH, *the ſine of any intermediate arc* KH, *whoſe exceſs above the leſſer arc* KB *is any given number of minutes.*

From B draw BM at right angles to EF, meeting GH in L; therefore [34. 1.] AB, GL, EM, are equal to one another; and becauſe EF and AB, that is, EM, are each of them given, their difference FM is given. Again, becauſe the points B, F, are very near one another, the circumferences FB, BH, may be conſidered as ſtraight lines; wherefore [4. 6.] FB is to BH, as FM to HL; but the circumferences FB, BH, and the ſtraight line MF, are each of them given; therefore HL is given, and AB, that is, GL, is given; therefore the whole ſtraight line GH, which is the ſine of the arc KH, is given. Which was to be found.

PROP. XII. PROB. *Fig.* 10.

27. *Let the arc* KBH *be one minute or ſixty ſeconds, and let its ſine* HG *be given, to find, nearly,* AB, *the ſine of* KB, *an arc of one or more ſeconds.*

Becauſe the arcs KH and KB are very ſmall, they may be conſidered as ſtraight lines; wherefore [4. 6.] KH is to KB, as HG to BA; but KH, KB, HG, are each of them given; therefore BA is given. Which was to be found.

PROP.

PROP. XIII. THEOR. *Fig.* 11.

28. *The rectangle contained by radius, and half the ſum of the ſines of two unequal arcs* GH *and* GHL, *is equal to the rectangle contained by the ſine of half their ſum, and the coſine of half their difference.*

Let C be the centre of the circle, and draw the diameter GA; join LH; and from the centre C, draw CO at right angles to LH, meeting the circumference of the circle in K, which therefore biſects LH in O, and the arc LKH in K, [3. 3. 4. 1. & 28. 3.]; from L, draw LD, at right angles to GA; and from O, K, H, draw OP, KE, HF, parallel to LD; laſtly, from O, H, draw OM, HN, parallel to GA.

HF and LD are the ſines of the arcs GH and GHL; and becauſe their difference HKL is biſected in K, the arc GHK is half of their ſum, and KE is its ſine; alſo CO is the coſine of half of their difference; and becauſe LO is equal to OH, LM is [2. 6.] equal to MN; therefore MD or OP is half the ſum of LD and ND or HF. And becauſe OP is parallel to KE, [4. 6.] CK is to KE, as CO to OP; therefore [16. 6.] the rectangle contained by CK and OP is equal to the rectangle contained by KE and CO. Q. E. D.

Cor. Hence, if KE, the ſine of half the ſum of two arcs GH, GHL, and CO, the coſine of half their difference, be each of them given, OP or MD, which is half the ſum of the ſines of the ſame arcs, is likewiſe given. For becauſe the rectangle contained by CK and OP is equal to the rectangle contained by KE and CO; [14.6.] CK is to KE, as CO to OP or MD; and if FH, that is, DN, the ſine of GH, one of the arcs, be given, DL, the ſine of the other arc GHL, is found by adding to OP or MD its exceſs above FH or DN. In like manner, if DL, the ſine of GHL, be given, DN or FH, the ſine of the other arc GH is found, by ſubtracting from OP or MD the exceſs of LD above it.

PROP. XIV. THEOR. *Fig.* 11.

29. *The ſine of any arc* GHL, *greater than ſixty degrees, but not greater than ninety, is equal to the ſine of another arc* GH, *as much leſs than ſixty degrees, together with the*

the ſine of its exceſs above ſixty degrees, and e contra.

Let GHK be an arc of ſixty degrees, other things remaining as in the preceding propoſition; let CK meet DL in Q; it is to be ſhown that DL is equal to FH together with LO. Becauſe the triangle LOM is [8. 6.] ſimilar to the triangle MOQ, that is (becauſe OM and DC are parallel) to the triangle CQD or CKE; the triangle LOM is [21. 6.] ſimilar to the triangle CKE; therefore [4. 6.] CK is to CE, as LO to LM: but becauſe KE is the ſine of ſixty degrees, CE, [def. 6. of this] the ſine of thirty degrees, is [1. of this] half of CK; therefore [D. 5.] LM is half of LO; and LN is [13. of this] equal to LO. Q. E. D.

Cor. Hence, if the ſine of each degree and minute, &c. to ſixty degrees be given, the ſine of each degree and minute, &c. above ſixty to ninety, may be found by adding to the ſine of ſixty, the ſine of the exceſs of the given degree above ſixty.

PROP. XV. PROB.

30. *The ſines of* 30, 15, 45, *and* 36 *degrees being each of them found by problems*

blems 1, 3, 5, 9 *and its cor. to conſtruct a table of ſines for every degree, minute, and ſecond, in the quadrant.*

(1) The ſine of 36° being found [cor. 9. of this], the ſines of 18°, 9°, 4° 30′, 2° 15′ are [3. of this] found: then [2. of this] the ſines of 54°, 72°, 81°, 85° 30′, 87° 45′, are found: alſo [3. of this] the ſines of 27°, 13° 30′, 6° 45′; 40° 30′, 20° 15′; 42° 45′, are found. Hence [2. of this] the ſines of 63°, 76° 30′, 83° 15′, 49° 30′, 69° 45′, 47° 15′, are found; and [3. of this] the ſines of 31° 30′, 15° 45′; 38° 15′; 24° 45′, are found. Hence [2. of this] the ſines of 58° 30′, 74° 15′, 51° 45′, 65° 15′, are found; and [3. of this] the ſine of 29° 15′; and [2. of this] its coſine, viz. 60° 45′ are found.

(2) The ſine of 45° being [5. of this] found, the ſines of 22° 30′, 11° 15′, are [3. of this] found; as alſo [2. of this] the ſines of 67° 30′, 78° 45′; and [3. of this] of 33° 45′, and [2. of this] of 56° 15′, are found.

(3) The ſines of 30° and 54° being each of them given, the ſine of 12° is [10. of this] found.

(4) The ſines of 12° being given, the ſines of 6°, 3°, 1° 30′, 45′, are [3. of this] found; and [2. of this] the ſines of 78°, 84°, 87°, 88° 30′, 89° 15′, are found; and [3. of this] the ſines of 39°, 19° 30′, 9° 45′; 42°, 21°, 10° 30′, 5° 15′; 43° 30′, 21° 45′; 44° 15′, are found; as alſo [2. of this] the ſines of 51°, 70° 30′, 80° 15′, 48°, 69°, 79° 30′, 84° 45′, 46° 30′, 68° 15′, 45° 45′, are found; and [3. of this] the ſines of 25° 30′, 12° 45′; 35° 15′; 24°; 34° 30′, 17° 15′; 39° 45; 23° 15′, are found: and hence [2. of this] the ſines 64° 30′, 77° 15′, 54° 45′, 66°, 55° 30′, 72° 45′, 50° 15′, 66° 45′, are found: hence alſo [3. of this] the ſines of 32° 15′; 33°, 16° 30′, 8° 15′; 27° 45′, are found; and [2. of this] likewiſe the ſines of 57° 45′, 57°, 73° 30′, 81° 45′, 62° 15′, are found; and [3. of this] the ſines of 28° 30′, 14° 15′; 36° 45′, are found; and [2. of this] the ſines of 61° 30′, 75° 45′, 53° 45′, are found; and [3. of this] the ſine of 30° 45′ is found; and [2. of this] the ſine of 59° 15′ is found.

(5) The ſine of 15° being [1. & 3. of this] given, the ſines of 7° 30′, 3° 45′, are [3. of this] found; and [2. of this] the ſines

of 75°, 82° 30′, 86° 15′, are found; and [3. of this] the ſines of 37° 30′, 18° 45′; 41° 15′, are found; and [2. of this] the ſines of 52° 30′, 71° 15′, 48° 45′, are found; and [3. of this] the ſine of 26° 15′; as alſo [2. of this] the ſine of 63° 45′, is found.

The number of ſines thus found is 120; and when they are all ſet down in order, the difference between any one of them, and the next preceding or following, is 45′. See the following table.

The intermediate ſines, to ſingle minutes and ſeconds, &c. are [11. & 12. of this] found. The calculation of ſines, however, is greatly abridged by the 13th and 14th propoſitions, with their corollaries.

The TABLE.

1	0°	45′	41	30°	45′	81	60°	45′
2	1	30	42	31	30	82	61	30
3	2	15	43	32	15	83	62	15
4	3	0	44	33	0	84	63	0
5	3	45	45	33	45	85	63	45
6	4	30	46	34	30	86	64	30
7	5	15	47	35	15	87	65	15
8	6	0	48	36	0	88	66	0
9	6	45	49	36	45	89	66	45
10	7	30	50	37	30	90	67	30
11	8	15	51	38	15	91	68	15
12	9	0	52	39	0	92	69	0
13	9	45	53	39	45	93	69	45
14	10	30	54	40	30	94	70	30
15	11	15	55	41	15	95	71	15
16	12	0	56	42	0	96	72	0
17	12	45	57	42	45	97	72	45
18	13	30	58	43	30	98	73	30
19	14	15	59	44	15	99	74	15
20	15	0	60	45	0	100	75	0
21	15	45	61	45	45	101	75	45
22	16	30	62	46	30	102	76	30
23	17	15	63	47	15	103	77	15
24	18	0	64	48	0	104	78	0
25	18	45	65	48	45	105	78	45
26	19	30	66	49	30	106	79	30
27	20	15	67	50	15	107	80	15
28	21	0	68	51	0	108	81	0
29	21	45	69	51	45	109	81	45
30	22	30	70	52	30	110	82	30
31	23	15	71	53	15	111	83	15
32	24	0	72	54	0	112	84	0
33	24	45	73	54	45	113	84	45
34	25	30	74	55	30	114	85	30
35	26	15	75	56	15	115	86	15
36	27	0	76	57	0	116	87	0
37	27	45	77	57	45	117	87	45
38	28	30	78	58	30	118	88	30
39	29	15	79	59	15	119	89	15
40	30	0	80	60	0	120	90	0

PROP. XVI. PROB. *Fig.* 5.

31. *The ſine* DH, *and the coſine* CH *of any arc* DB *being each of them given, to find* BE *its tangent, and* CE *its ſecant.*

Becauſe DH and EB are each of them at right angles to CB, they are [28. 1.] parallel; therefore [4. 6.] CH is to HD, as CB to BE; that is, the coſine of the arc DB is to its ſine, as radius to its tangent: but the three firſt of theſe proportionals are each of them given; therefore the fourth BE is alſo given.

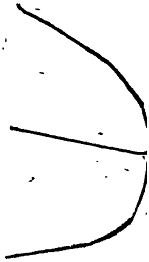
Likewiſe CH is to CB, as CD or CB to CE; that is, the ſecant is a third proportional to the coſine and radius; wherefore the ſecant CE is given.

In like manner, the tangent and ſecant of any other arc is found. Which was to be done.

32. The table of natural verſed ſines to ninety degrees, is conſtructed by ſubtracting the natural coſine of the given degree or minute from radius. Thus, if 9998477, which is the natural ſine of 89°, or coſine of 1°, be ſubtracted from radius 10000000, the

M
L
H
a

G



the remainder 1523 ſhall be the natural verſed ſine of one degree; and ſo on.

SECT. II.

Method of conſtructing a table of Logarithms.

1. IF four magnitudes be proportionals, and if the firſt be a multiple or part of the ſecond, the third is the ſame multiple, or the ſame part of the fourth. [Prop. D. 5. Simſon's Euclid.]

2. Hence, if a ſeries of numbers decreaſe or increaſe, ſo that the quotient ariſing by dividing the firſt term by the ſecond, be equal to the quotient ariſing by dividing the ſecond term by the third, and to that ariſing by dividing the third by the fourth, &c. it is ſaid to be a ſeries in geometrical proportion continued. Such is the ſeries 32, 16, 8, 4, 2, 1, &c. or 1, 10, 100, 1000, 10000, &c.

3. A ſeries of numbers decreaſing or increaſing, ſo that the difference between the firſt and ſecond terms is equal to the difference between the ſecond and third, and to that between the third and fourth, and ſo on, is ſaid to be in arithmetical pro-

portion; as, 5, 4, 3, 2, 1, 0, &c. or 1, 3, 5, 7, 9, &c.

PROP. I. THEOR. *Plate* 3. *fig.* 1.

4. *If three magnitudes* A, BF, CH, *be arithmetical proportionals, the ſum of the two extremes* A *and* CH *is equal to twice the middle term* BF.

Make BE equal to A, and CG equal to BF; therefore GH is equal to EF. Becauſe CG is equal to BF, and GH to EF, CH is equal to BF and FE together; to each of theſe equals add the equals A and BE, then CH and A together ſhall be equal to twice BF. Q. E. D.

PROP. II. THEOR. *Fig.* 1.

5. *If four magnitudes* A, BF, CH, DL, *be arithmetical proportionals, the ſum of the two extremes* A *and* DL *is equal to the ſum of the two means* BF *and* CH.

Conſtruct as in the preceding, and make DK equal to CH; KL is equal to GH. Becauſe [by the preceding] CH, that is, DK and A are together equal to twice BF, that is, to BF and CG; and that KL is equal to GH; DL and A are together equal to BF and CH together. Q. E. D.

In like manner it is demonſtrated, that if any number of magnitudes be arithmetical proportionals, the ſum of the two extremes is equal to the ſum of any two terms equally diſtant from the extremes, and if there be an odd number of terms to twice the middle term.

6. If the upper ſeries, in the two lines ſubjoined, be in continued geometrical, and the lower in arithmetical proportion, the ſeveral terms of the arithmetical ſeries are called the logarithms of the correſponding terms of the geometrical ſeries.

1, 2, 4, 8, 16, 32, 64, 128, 256, 512,
0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

7. If the geometrical ſeries begin at unit, and the arithmetical at nothing, as in this example, the ſeveral terms of the arithmetical ſeries will ſhew the diſtance of the correſponding terms in the geometrical ſeries from unit. Thus, 6 in the arithmetical ſeries ſhews that the correſponding term 64, in the geometrical ſeries, is in the ſixth place from unit.

8. If any number of terms of the arith-

metical ſeries be added together, the ſum is the index or logarithm of the product of the correſponding terms of the geometrical ſeries. Thus, 2 and 3, or 5, is the logarithm of 4 times 8, or 32; 3 and 6, or 9, is the logarithm of 8 times 64, or 512, &c.; 2 and 3 and 4, or 9, is the logarithm of 4 times 8, multiplied by 16, viz. 512. Hence it is that multiplication of numbers is performed by the addition of their logarithms.

9. The difference between any two terms of the arithmetical ſeries is the logarithm of the quotient of the correſponding terms of the geometrical ſeries. Thus 2, the difference between 6 and 4, is the logarithm of 4, which is the quotient of 64 divided by 16. Hence diviſion of numbers is performed by the ſubtraction of their logarithms.

10. Becauſe any number, except unit, its ſquare, its cube, its fourth power, &c. make a ſeries in geometrical proportion; as, 2, 4, 8, 16, in the preſent example; twice the logarithm of the number is the logarithm of its ſquare; three times the logarithm of the number is the logarithm of its cube, &c. [art. 8.]. Thus, twice 2, or 4,

is the logarithm of 16, which is the ſquare of 4; three times 2, or 6, is the logarithm of 64, which is the cube of 4.

11. For the ſame reaſon, half the logarithm of a number is the logarithm of its ſquare root; the third part of the logarithm of a number is the logarithm of its cube root, &c. [art. 9.]. Thus, the half of 4, or 2, is the logarithm of 4, which is the ſquare root of 16; the third part of 9, or 3, is the logarithm of 8, which is the cube root of 512, &c.

12. Though a geometrical mean proportional be placed between each two terms of the geometrical ſeries, the new ſeries will alſo be in continued geometrical proportion. In like manner, though a geometrical mean proportional be placed between each two terms of this laſt ſeries, the ſeries thence ariſing will ſtill be in geometrical proportion continued; and ſo on.

13. Whatever number of geometrical mean terms is inſerted between each two terms of the geometrical ſeries, if the ſame number of arithmetical mean terms be inſerted between each two terms of the arithmetical ſeries, it is manifeſt, that the ſeries will remain in arithmetical proportion, and

that each of its terms is the logarithm or index of each correſponding term of the geometrical ſeries as formerly.

In this manner a table of logarithms is conſtructed.

14. A geometrical mean proportional is found between two numbers [17. 6.] by extracting the ſquare root of the product of the two numbers propoſed. Thus, a geometrical mean proportional between 4 and 8 is the neareſt ſquare root of 32.

An arithmetical mean proportional between two numbers is [1. of this] half the ſum of the two propoſed numbers: ſo the half of 5 is an arithmetical mean proportional between 2 and 3.

15. The imperfections of a table of logarithms are, that an exact mean proportional cannot always be found between each two of the terms of the ſeries, and that the difference between each two terms of the geometrical ſeries cannot be exactly unit. The firſt of theſe imperfections is remedied by dividing each term into a great number of decimal places; and the laſt, by inſerting ſo many terms, as that their difference may not much exceed unit.

16. A geometrical ſeries beginning at unit, and increaſing in a tenfold proportion; and an arithmetical ſeries beginning at nothing, and increaſing by unit, as in the margin, are found moſt convenient for conſtructing a table of logarithms.

1,	10,	100,	1000,	&c.
0,	1,	2,	3,	&c.

17. Hence the logarithm of 1 is 0, the logarithm of 10 is 1, that of 100 is 2, and that of 1000 is 3, &c.; therefore the logarithm of each intermediate term between 1 and 10 is a proper fraction; the logarithm of each term between 10 and 100 is unit with a proper fraction, &c.

18. It is manifeſt that the logarithms of each of the intermediate terms between 1 and 10, 10 and 100, &c. of the geometrical ſeries, cannot [art. 14.] be found exactly; ſuch as the logarithm of 2, of 3, of 4, &c.; alſo that of 11, of 12, of 13, &c.; but to find them to ſuch a degree of exactneſs as anſwers any uſe, each term is to be divided into many decimal parts, as follows.

1.0000000, 10.0000000, 100.0000000, &c.
0.00000000, 1.00000000, 2.00000000, &c.

19. In order to find the logarithm of each whole number between 1 and 10,

viz. the logarithm of 2, of 3, &c. as many geometrical mean terms are to be found between 1 and 10, as there are whole numbers between them; and theſe mean terms muſt be ſuch, that the difference between any one of them, and the next preceding or following, is very near unit. Arithmetical mean terms are to be found between 0 and 1, each of them anſwering to each of the geometrical mean terms. In like manner, geometrical mean terms are to be found between 10 and 100, and as many arithmetical mean terms between 1 and 2, &c.

20. To find the logarithm of 9.

Between 1 and 10, find a geometrical mean proportional; that is, [art. 14.] the neareſt ſquare root of 10.0000000, viz. 3.1622777.

Between 0.00000000, which is the logarithm of 1.0000000, and 1.00000000, which is the logarithm of 10.0000000, find an arithmetical mean proportional; that is, [art. 14.] the half of 1.00000000, viz. 0.50000000; this laſt number is the logarithm of the geometrical mean 3.1622777. But becauſe 3.1622777 is much leſs than 9.0000000, another geometrical mean muſt be found between 3.1622777 and 10.0000000, viz. 5.6234132; and then an arithmetical mean between the former arithmetical mean 0.50000000 and 1.00000000, viz. 0.75000000. This laſt number is the logarithm of 5.6234132, but is ſtill far from the logarithm of 9.0000000; becauſe 5.6234132 is much leſs than 9.0000000; and therefore another geometrical mean muſt be found between 5.6234132 and 10.0000000 with its correſponding logarithm, and ſo on, till a geometrical mean is found, whoſe difference from 9.0000000 may be neglected. An arithmetical mean is to be found as often; and the arithmetical mean, correſponding to the geometrical mean that comes neareſt to 9.0000000, is the logarithm of 9.

21. It will be found, on trial, that the fifth geometrical mean, found as above, is 9.3057204, which is greater than 9.0000000; therefore the next geometrical mean muſt be found between 9.3057204 and the next preceding geometrical mean 8.6596432. The geometrical mean between theſe laſt is 8.9768713, which is leſs than 9.0000000. The next following

is to be found between 9.3057204 and 8.9768713, which will be greater than 9.0000000; and the next again is to be found between this laſt one and the next preceding; and ſo on.

22. No fewer than twenty-ſix geometrical mean terms, and as many arithmetical, are neceſſary, in order to find the logarithm of 9, by this method, tolerably exact, as may be ſeen by the following table.

The TABLE.

	Mean Prop.	Logarithms		Mean Prop.	Logarithms.
1.	1.0000000	0.00000000	14.	9.0021388	0.95434579
	3.1622777	0.50000000		9.0008737	0.95428467
	10.0000000	1.00000000		8.9996088	0.95422363
2.	10.0000000	1.00000000	15.	9.0008737	0.95428467
	5.6234132	0.75000000		9.0002412	0.95425415
	3.1622777	0.50000000		8.9996088	0.95422363
3.	10.0000000	1.00000000	16.	9.0002412	0.95425415
	7.4989421	0.87500000		8.9999250	0.95421889
	5.6234132	0.75000000		8.9996088	0.95422363
4.	10.0000000	1.00000000	17.	9.0002412	0.95425415
	8.6596432	0.93750000		9.0000831	0.95424652
	7.4989421	0.87500000		8.9999250	0.95423889
5.	10.0000000	1.00000000	18.	9.0000831	0.95424652
	9.3057204	0.96875000		9.0000041	0.95424271
	8.6596432	0.93750000		8.9999250	0.95423889
6.	9.3057204	0.96875000	19.	9.0000041	0.95424271
	8.9768713	0.95312500		8.9999650	0.95424080
	8.6596432	0.93750000		8.9999250	0.95423889
7.	9.3057204	0.96875000	20.	9.0000041	0.95424271
	9.1398170	0.96093750		8.9999845	0.95424217
	8.9768713	0.95312500		8.9999650	0.95424080
8.	9.1398170	0.96093750	21.	9.0000041	0.95424271
	9.0579777	0.95703125		8.9999943	0.95424223
	8.9768713	0.95312500		8.9999845	0.95424217
9.	9.0579777	0.95703125	22.	9.0000041	0.95424271
	9.0173333	0.95507812		8.9999992	0.95424247
	8.9768713	0.95312500		8.9999943	0.95424223
10.	9.0173333	0.95507812	23.	9.0000041	0.95424271
	8.9970796	0.95410156		9.0000016	0.95424259
	8.9768713	0.95312500		8.9999992	0.95424247
11.	9.0173333	0.95507812	24	9.0000016	0.95424259
	9.0072008	0.95458984		9.0000004	0.95424253
	8.9970796	0.95410156		8.9999992	0.95424247
12.	9.0072008	0.95458984	25.	9.0000004	0.95424253
	9.0021388	0.95434570		8.9999998	0.95424250
	8.9970796	0.95410156		8.9999992	0.95424247
13.	9.0021388	0.95434570	26.	9.0000004	0.95442253
	8.9996088	0.95422363		9.0000000	0.95424251
	8.9970706	0.95410156		8.9999998	0.95424250

23. The logarithms of 8, 7, 6, &c. may be found in the ſame manner. The logarithm of every number, however, does not require ſo tedious an operation.

Becauſe multiplication of numbers is [art. 8.] performed by the addition of their logarithms, and [art. 9.] diviſion by ſubtraction, the ſum of the logarithms of any two numbers is the logarithm of their product; and the difference between the logarithms of two numbers, is the logarithm of their quotient. Thus the ſum of the logarithms of 2 and 3 is the logarithm of 6; the ſum of the logarithms of 3 and 4 is the logarithm of 12, &c. the difference between the logarithms of 8 and 2 is the logarithm of 4, the difference between the logarithms of 15 and 3 is the logarithm of 5, &c.

And becauſe involution [art. 10.] is performed by doubling, tripling, &c. the logarithms of the roots and evolution [art. 11.] by finding the half, the third part, &c. of the logarithm of the power; twice the logarithm of 2 is the logarithm of 4, viz. the ſquare of 2; three times the logarithm of 2 is the logarithm of 8, viz. the cube of 2, &c.; the half of the logarithm of 9 is the logarithm of 3; the third part of the logarithm of 64 is the logarithm of 4; and ſo on. For a more particular account of the conſtruction of logarithms, and the various methods by which the operations are ſhortened, I refer to the authors who have wrote on that ſubject on purpoſe.

24. Becauſe the difference of the logarithms of two numbers is the logarithm of their quotient, the logarithm of a fraction is the difference of the logarithms of the numerator and denominator. Hence, becauſe in the common logarithms above deſcribed, the logarithm of 1 is 0, the logarithm of a proper fraction is negative, or leſs than 0. Thus the logarithm of the proper fraction $\frac{3}{8}$ is 0.4259687, viz. the difference of the logarithms of 3 and 8, which is leſs than 0 or negative, becauſe the diviſor 8 is greater than the dividend 3, and that the logarithm of 1 is 0. On the other hand, the logarithm of the improper fraction $\frac{8}{3}$ is alſo 0.4259687; but poſitive, or greater than 0, becauſe 8 is greater than 3, and that the logarithm of 1 is 0. Negative logarithms are as uſeful in the doctrine

of proper fractions as poſitive, in that of whole or mixt numbers.

25. For the ſame reaſons, the logarithm of a mixt number is the difference of the logarithms of the numerator and denominator of the mixt number reduced to an improper fraction. Thus $3\frac{4}{5}$ reduced to an improper fraction is $\frac{19}{5}$, whoſe logarithm is 0.5797836, viz. the difference of the logarithms of 19 and 5.

26. The logarithms are, in the table, found oppoſite to their reſpective numbers from unit, ſo far as the table reaches. And becauſe the logarithms of the numbers 1, 10, 100, 1000, &c. are 0, 1, 2, 3, &c. only the decimal part of the logarithms of the intermediate numbers are commonly ſet down; particularly in Sherwin's tables.

27. If a logarithm, different from any in the table, be given, it is the logarithm of a mixt number, between the numbers anſwering to the logarithms next leſs and greater than it. Thus the logarithm 0.5797836 is greater than the logarithm of 3, but leſs than that of 4; and therefore it is the logarithm of ſome mixt number between 3 and 4.

28. To find the logarithm of a proper fraction $\frac{3}{8}$. Subtract the logarithm of 3 from the logarithm of 8, the remainder; viz. the negative logarithm 0.4259687, [art. 24.] is the logarithm of $\frac{3}{8}$. But becauſe 0, 1, 2, 3, &c. are the logarithms of the numbers 1, 10, 100, 1000, &c. it is more convenient firſt to reduce a vulgar fraction to a decimal, and then to find the logarithm of the decimal as a whole number. Thus $\frac{3}{8}$, reduced to a decimal fraction, is .375. The logarithm of .375, conſidered as a whole number, is 2.5740313; and the denominator of the decimal fraction .375 is 1000, whoſe logarithm is 3.0000000. The difference between theſe two logarithms is 0.4259687; the ſame as before.

Hence, if a negative logarithm 0.4259687 be given, the proper fraction correſponding to it will be found by ſubtracting 0.4259687 from 3.0000000, the remainder 2.5740313 is the logarithm of the numerator of the decimal fraction, viz. .375, which is equal to $\frac{3}{8}$.

29. To find the logarithm of a mixt number 65.7. Find the logarithm of 657 as a whole number, which is 2.8175654;

and from its index 2, ſubtract 1, which is the logarithm of 10, viz. the denominator of the decimal part 7; the remainder 1.8175654 is the logarithm of the mixt number 65.7.

Hence the mixt number anſwering to a logarithm 1.8175654 is found, by finding the number that anſwers to its decimal part, without regard to the index, which is 657; and dividing 657 by 10, the quotient 65.7 is the mixt number required. Likewiſe the mixt number anſwering to the logarithm 0.5797836 is found, by finding the number anſwering to .5797836, viz. the decimal part of the given logarithm, which is 380: and becauſe the index of the given logarithm is 0, and that the index of the logarithm anſwering to 380 is 2, if 380 be divided by 100, the quotient 3.80 is the mixt number ſought, which is equal to $3\frac{4}{5}$.

30. To find the logarithm of a number exceeding any in the table, or which is greater than the table extends to.

Example. To find the logarithm of 64856 by means of a table, not exceeding 10000. Divide 64856 by 10, the quotient is 6485.6; ſubtract 3.8119100, which is the logarithm of 6485; from 3.8119769, which is the logarithm of the next greater number 6486, the remainder is 669. To the three numbers 10, 669, and 6, viz. the laſt figure of the propoſed number, find a fourth proportional 401. This laſt number being added to the decimal part of the logarithm of 6485, and unit to its index, gives 4.8119501, which is nearly the logarithm of 64856. This method is fully exemplified in the treatiſe prefixed to Sherwin's excellent tables; to which I refer the reader. It proceeds upon the ſuppoſition, that numbers are to one another as their logarithms. This ſuppoſition, however, is not ſtrictly true, as will be ſhewn afterwards; but it anſwers common uſes in finding the logarithms of great numbers.

31. To find nearly the number anſwering to the logarithm 6.9934362, greater than any in the table. When the table is not calculated for numbers exceeding 10000, ſubtract 4.0000000, which is the logarithm of 10000, from the given logarithm 6.9934362, and find in the table 985, which is the number anſwering to their difference 2.9934362; then multiply 985 by 10000, the product 9850000 is

nearly the number anſwering to the logarithm 6.9934362.

Some uſes of Logarithms.

32. To multiply numbers by the logarithms. Add the logarithm of the multiplier to that of the multiplicand, the ſum ſhall be the logarithm of the product.

Example. To multiply 339 by 264. The logarithm of 264 is 2.4216039, that of 339 is 2.5301997; the ſum of theſe two logarithms is 4.9518036, which is the logarithm of the product 89496.

33. To divide numbers by the logarithms. Subtract the logarithm of the diviſor from that of the dividend, the remainder ſhall be the logarithm of the quotient.

Example. To divide 768 by 24. From 2.8853612, which is the logarithm of 768, ſubtract 1.3802112, which is the logarithm of 24; the remainder 1.5051500 is the logarithm of the quotient 32.

Proportion by the Logarithms.

34. Add the logarithms of the ſecond and third terms together, and from the ſum ſubtract the logarithm of the firſt term, the remainder ſhall be the logarithm of the fourth proportional.

Example 1. To find a fourth proportional to the three numbers 4, 8, 16. Add 0.9030900, which is the logarithm of 8, to 1.2041200, which is the logarithm of 16, and from the ſum 2.1072100, ſubtract 0.6020600, which is the logarithm of 4, the remainder 1.5051500 is the logarithm of the fourth proportional 32.

Example 2. To find a third proportional to two numbers, 12 and 36. From twice the logarithm of 36, which is 3.1126050, ſubtract 1.0791812, which is the logarithm of 12, the remainder 2.0334238 is the logarithm of the third proportional 108.

35. To raiſe numbers to their powers by the logarithms.

Twice the logarithm of a number is the logarithm of its ſquare; three times its logarithm, is the logarithm of its cube, &c.

Example. 1.2041200, which is twice

the logarithm of 4, is the logarithm of 16, which is the ſquare of 4. 1.8061800, which is three times the logarithm of 4, is the logarithm of 64, which is the cube of 4.

36. To extract roots by the logarithms.

The half of the logarithm of a number is the logarithm of its ſquare root; the third part of the logarithm of a number is the logarithm of its cube root, &c.

Example. 0.6989700, the half of 1.3979400, which is the logarithm of 25, is the logarithm of 5, which is the ſquare root of 25. 0.4771213, the third part of 1.4313638, which is the logarithm of 27, is the logarithm of 3, which is the cube root of 27, &c.

Of the Hyperbolical Logarithms.

37. The logarithms deſcribed in the preceding part of this ſection are found moſt convenient for the more common calculations, and have been long in uſe. They are ſomething different, however, from thoſe firſt publiſhed by the illuſtrious inventor, Baron Napier. This memorable diſcovery of that diſtinguiſhed promoter of

ſcience may be deſcribed in the following manner, from his *Mirifici Logar. Canonis Deſcriptio*, firſt publiſhed at Edinburgh in 1614.

38. Let PAF, plate 3. fig. 2. be any ſtraight line terminated in P, but unlimited towards F. Let a point P be ſuppoſed to move along this line from P towards F, with a velocity always proportional to its diſtance from the fixed extremity P; that is, let the velocity of the moving point in B be to its velocity in A, as PB to PA; and the velocity in C to the velocity in B, as PC to PB, or as PB to PA, &c. Hence PA, PB, PC, PD, &c. are in continued geometrical proportion. Likewiſe AB, BC, CD, &c. are in continued geometrical proportion. For, becauſe PC is to PB, as PB to PA; by diviſion, BC is to PB, as AB to to PA; alternately, BC is to AB, as PB to PA; and, by inverſion, AB is to BC, as PA to PB. In like manner, BC may be ſhewn to be to CD, as PB to PC; that is, as PA to PB, or as AB to BC, as has been ſhewn; and ſo on. And becauſe the velocities are proportional to the ſpaces deſcribed, the ſpaces AB, BC, CD, &c. ſhall be deſcribed in equal times.

39. Let the velocity of the point deſcribing the line PAF in this manner, in any point of the line A, be called V; and, with this velocity remaining conſtant, let another point deſcribe the ſpace A′B′ upon the line e′A′F′, in the ſame time that AB is deſcribed upon PF, with the accelerated velocity ſuppoſed above. There are therefore two points, one of them moving from A towards F, with an accelerated velocity; and deſcribing the proportional ſpaces AB, BC, CD, &c. in equal times; the other moving from A′ towards F′ uniformily, deſcribing the equal ſpaces A′B′, B′C′, C′D′, &c. in equal times, viz. in the times that AB, BC, CD, &c. are deſcribed. Hence, while PB, PC, PD, &c. are in continued geometrical proportion, A′B′, A′C′, A′D′, &c. are in arithmetical proportion; and therefore A′B′, A′C′, A′D′, &c. are the logarithms of PB, PC, PD, each of each.

The logarithm of PS, any part of the diſtance deſcribed by the accelerated velocity, is expreſſed by A′S′, deſcribed with the uniform velocity in the ſame time that AS has been deſcribed, the velocities being equal in A and A′; and ſo on.

40. Hence, if the two points deſcribing

the ſpaces PF and A′F′, the one with an accelerated, the other with an uniform motion, as above, depart from A and A′ at the ſame time, the logarithm of PA ſhall be 0, becauſe the point moving uniformly has not then deſcribed any ſpace.

41. If A′B′, A′C′, A′D′, &c. which expreſs the logarithms of PB, PC, PD, &c. be poſitive quantities, PA 1, and A′ 0, A′*b*′, the logarithm of any proper fraction P*b* ſhall be negative; that is, A′*b*′ ſhall be deſcribed with an uniform motion in the contrary direction. For the ſame reaſon, A′*c*′ is the logarithm of A*c*; and the logarithm of a proper fraction is equal to that of its denominator; but they have contrary ſigns: for if P*b* be $\frac{1}{2}$, PA 1, PB 2; and if A′B′ be the logarithm of PB 2, A′*b*′, which is equal to A′B′, ſhall be the logarithm of P*b*, which is $\frac{1}{2}$ taken in the contrary direction; or in numbers, with a contrary ſign: the logarithms of $\frac{1}{3}$, $\frac{1}{4}$, &c. are equal to the logarithms of 3, 4, &c. the ſigns being contrary.

A′B′, B′C′, C′D′, &c. may alſo be conſidered as the logarithms of any quantities proportional to PB, PC, PD, &c. as AB, BC, CD, &c.

42. The logarithm of any ratio, as that of PC to PB, or of the number expreſſing PC, divided by the number expreſſing PB, is B′C′ or A′B′, viz. the logarithm of the difference of PC, PB; and the logarithm of the product of the number expreſſing PC, multiplied by the number expreſſing PB, is A′C′, viz. the ſum of the logarithms of PB, PC, or of AB, BC, &c.

43. Becauſe PD, PC, PB, are proportionals; and when three magnitudes are proportionals, the firſt is to the third in the duplicate ratio of the firſt to the ſecond; that is, as the ſquare of the firſt to the ſquare of the ſecond, the logarithm of the ratio of the ſquare of PD to the ſquare of PC, (being the ſame with the logarithm of the ratio of PD to PB), is the logarithm of BD; that is, B′D′, or twice B′C′, which is the logarithm of BC or PC. In like manner, becauſe PE, PD, PC, PB, are continued proportionals; and when four magnitudes are continued proportionals, the firſt is to the fourth in the triplicate ratio of the firſt to the ſecond; that is, as the cube of the firſt to the cube of the ſecond; the logarithm of the ratio of the cube of PE to the cube of PD, (being the ſame with the

logarithm of the ratio of PE to PB), is the logarithm of BE; that is, B′E′, or three times B′C′, which is the logarithm of BC or PC; and ſo on, for any other power.

44. The next thing to be conſidered is, How Napier found each number exhibiting the logarithm of each term of the ſeries of natural numbers 1, 2, 3, 4, 5, &c.

In order to this, he ſuppoſes PA unit expreſſed thus, 1.0000000, and A′ the point where the uniform motion begins; conſequently the logarithm of unit is 0.0000000, &c. *in infinitum.* Let PB be 2, or 2.0000000, it is required to find the number exhibiting its logarithm A′B′. Let ſo great a number of geometrical mean proportionals be ſuppoſed between PA and PB, that A*a*, the exceſs of the leaſt of them above PA unit, may be conſidered as deſcribed with an uniform motion. Hence, according to Napier, A*a* itſelf may be conſidered as the logarithm of P*a*; and ſuppoſing P*a* to be 1.0000001, its logarithm is 0.0000001. Then he calculates what is the number of geometrical mean proportionals, ſuch as, P*a*, between PA 1.0000000 and PB 2.0000000; that is, the number of geometrical mean proportionals between

1.0000000 and 2.0000000, ſuch that the exceſs of each of them above the next preceding is equal to *Aa* 0.0000001, or very nearly ſo, and finds that there are 6931472 of ſuch geometrical mean proportionals between PA and PB. He therefore multiplies *Aa* 0.0000001 by 6931472, and the product 0.6931472 is A'B' the logarithm of 2. Proceeding in the ſame manner, he finds, that there are 23025850 ſuch geometrical mean proportionals, as above, between 1 and 10. According to this hypotheſis, therefore, 2.3025850 is the logarithm of 10.

45. The Noble author afterwards obſerved, that it would be more convenient to have unit for the logarithm of ten, two for the logarithm of an hundred, three for the logarithm of a thouſand, &c. as in the common tables, and recommended to Mr Briggs to alter the form of the logarithms. This Mr Briggs undertook and executed. Hence Napier's firſt logarithms are to thoſe calculated by Mr Briggs, which are commonly uſed, as 2.3025850 to 1; and the one kind can eaſily be reduced to the other.

46. Several of the diſcoveries in the

higher geometry that followed that of the logarithms, contributed much to the finding the logarithms of numbers with greater eaſe. I ſhall mention a few of theſe in their order.

In order to underſtand this and the three following articles, the reader is ſuppoſed to know that the rectangles AEM, fig. 3, AFK, ABG, are equal to one another, [12. 2. Appoll. Con. or 16. 3. Simſ.].

Gregory of St Vincent obſerved, that if AE, AF, AB, &c. be taken in continued geometrical proportion from the centre A, upon the aſymptote AB of a hyperbola GKM, and if from the extremities of theſe proportionals E, F, B, &c. ſtraight lines be drawn parallel to the other aſymptote AC, meeting the hyperbola in M, K, G, &c. the ſpaces EMKF, FKGB, &c. ſhall be equal to one another; as alſo the ſectors AMK, AKG. For becauſe the rectangle AEM is equal to the rectangle AFK, the triangle AEM is equal to the triangle AFK; and if the common triangle AEO be taken from both, the triangle AOM ſhall be equal to the trapezium EFKO; and if the ſpace KOM be added to each of theſe equals, the ſector AMK ſhall be equal to the ſpace EMKF. In like manner, the ſector AKG is ſhewn to be equal to the ſpace FKGB. Draw KN, GQ, parallel to AB, meeting EM, FK in P, L. And becauſe AE, AF, AB, are proportional to BG, FK, EM; BF, FE, or GL, KP, are proportional to EP, FL, and to PM, LK; therefore [16. 6.] GBFL is equal to KFEP and the rectilineal triangle GLK equal to the rectilineal triangle KPM: and if an indefinite number of geometrical mean proportionals be found between AE, AF, and between AF, AB, and from their extremities, parallels to AC be drawn, meeting the hyperbola, the rectangle contained by the difference of any two terms, taken on the aſymptote, and by the difference of the two correſponding terms of the ordinates, is equal to the rectangle contained by the difference of any other two terms taken on the aſymptote, and by the difference of the two correſponding terms of the ordinates; therefore the hyperbola GKM is the *locus* of theſe points; and the ſpace EMKF is equal to the ſpace FKGB; as alſo the ſector AMK to the ſector AKG. Q. E. D.

The ſame thing may be proved by the doctrine of fluxions. See Maclaurin's Fluxions,

xions, book 1. art. 177. Hence, if the rectangles FM, FR, be completed, the ſpaces TKM, RGK, ſhall be equal. Likewiſe, if CF, the ſquare of AF, and the rectangle DB, be completed, and if the diameters A*u*, AS, be drawn, the ſpaces *u*KM, SGK, ſhall be equal.

47. Hence, when EF, EB, &c. increaſe in geometrical proportion, the ſpaces EMKF, EMKGB, &c. increaſe in arithmetical proportion, and theſe ſpaces exhibit the logarithms of EF, EB. If now AE, AF, AB, &c. be to one another as the numbers 1, 2, 3, &c. or very nearly ſo, ſuppoſing the logarithm of AE to be 0; and if an infinite ſeries be found expreſſing the ſpace EMKF in parts of EMDA, which is the ſquare of AE unit, this ſeries ſhall be the logarithm of 2, according to this ſyſtem; and the ſeries expreſſing the ſpace EMKGB, ſhall be the logarithm of 3; and ſo on.

48. A ſeries of this kind was firſt diſcovered, in the following manner, by the excellent mathematician Viſcount Brounker. [Phil. Tranſ. N° 34. p. 645.

(1) Let DHK, fig. 4. be a hyperbola whoſe centre is A, and whoſe aſymptotes AC, AG,

are at right angles to one another; let D be the vertex of the tranſverſe axis, and draw DC, DE, parallel to the aſymptotes meeting them in C and E; AEDC is the ſquare of AE. Make EG equal to AE, and draw GK parallel to ED meeting the hyperbola in K. If AE be 1, its ſquare AEDC is 1; and the rectangle AGK is alſo 1; and becauſe AG is 2, GK or EP is $\frac{1}{2}$; wherefore EGKP is [1. 6.] one half of AD.

(2) Biſect GE in F, and draw FH parallel to ED meeting the hyperbola in H, and draw HQ parallel to AG, meeting ED in Q. And becauſe AF is to AE, as ED to FH or EQ; by converſion, AF is to FE, as ED to DQ; but FE is the third part of AF; therefore [D. 5.] DQ is the third part of DE or of 1; and if DQ be ſubtracted from DP, which is one half, the remainder QP ſhall be one ſixth part of DE; wherefore the right-angled parallelogram HP is contained by HQ $\frac{1}{2}$, and by QP $\frac{1}{6}$; and is therefore $\frac{1}{12}$ of AD.

(3) Biſect FE in L, and draw LM, MR, as before; and becauſe LE is $\frac{1}{5}$ of LA, DR is $\frac{1}{5}$ of DE 1; and if DR $\frac{1}{5}$, together with QP $\frac{1}{6}$, be ſubtracted from DP $\frac{1}{2}$, the remainder RQ ſhall be $\frac{8}{15}$; which being

multiplied by MR $\frac{1}{4}$, gives $\frac{1}{30}$ of AD for the rectangle MQ.

(4) Biſect FG in N, and complete the rectangle FO. Becauſe NF is $\frac{1}{7}$ of NA, HY is $\frac{1}{7}$ of HF, or of QF $\frac{2}{3}$: if therefore HY be ſubtracted from QP or HV, the remainder YV ſhall be $\frac{1}{14}$; which being multiplied by ZV $\frac{1}{4}$, gives $\frac{1}{56}$ of AD for the rectangle VO. In like manner, by biſecting EL, LF, FN, NG; RS, *u*T, YW, ZX, ſhall be found to be $\frac{1}{90}$, $\frac{1}{132}$, $\frac{1}{182}$, $\frac{1}{240}$, of AD; and this ſeries continued *in infinitum* exhauſts the hyperbolical ſpace EDHKG. Brouncker calculates a great number of the terms of this ſeries continued, in decimal parts, and finds the ſum of them to be .6931472 for the logarithm of 2: and when AG is to AE, that is, ED to GK, as 10 to 1, he finds the hyperbolical ſpace EDHKG to be 2.3025850, the ſame with Napier. Hence theſe logarithms are called hyperbolical. Not but that the common logarithms are alſo hyperbolical; for they expreſs the hyperbolical ſpace when the aſymptotes contain an angle of 54° 16′.

Mercator likewiſe diſcovered an infinite ſeries, expreſſing the hyperbolical ſpace above, by a different method: His ſeries is, $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, &c.; and if x be ſuppoſed leſs than unit, for example $\frac{1}{5}$, the ſeries will be $\frac{1}{5} - \frac{1}{2 \times 25} + \frac{1}{3 \times 125} - \frac{1}{4 \times 625}$, &c. If the ſum of this ſeries be taken in decimal fractions, the hyperbolical logarithms of numbers that do not much exceed unit are eaſily obtained; and by this means the logarithms of moſt whole numbers are found. For example, if twice the logarithm of $\frac{4}{3}$ be added to the logarithm of $\frac{9}{8}$, the ſum is the logarithm of 2; and if three times the logarithm of 2 be added to the logarithm of $\frac{5}{4}$, the ſum is the logarithm of 10, &c.

Many more methods of finding logarithms might be mentioned, were it neceſſary. The reader may conſult the treatiſe prefixed to Sherwin's Mathematical Tables.

50. The diſcovery of the logarithms gave riſe to the logarithmic curve. In the ſtraight line BH, fig. 5. let the points B, E, K, H, &c. be taken at equal diſtances from one another; from theſe points let ſtraight lines be drawn at right angles to BH in continued geometrical proportion; that is, BA to ED, as ED to KL, and as KL to HG, &c. The logarithmic curve

is that which paſſes through A, D, L, G, the other extremities of the geometrical proportionals, and alſo through the extremities of any number of geometrical mean proportionals placed at equal diſtances between BA and ED, and between ED and KL, &c.

While the geometrical mean proportionals or ordinates increaſe or decreaſe in continued geometrical proportion, the correſponding parts of BH, or abſciſſes, increaſe or decreaſe in arithmetical proportion; and therefore theſe laſt repreſent the logarithms of the former. Thus, if KL, ED, BA, &c. repreſent the numbers 1, 2, 3, &c. and if the logarithm of 1 be 0, KE, KB, &c, ſhall repreſent the logarithms of 2, 3, &c. The ordinates taken on the other ſide of KL, ſuch as HG, repreſent proper fractions: and becauſe the logarithm of KL repreſenting 1 is 0, if the ſtraight lines repreſenting the logarithms taken from K towards B be poſitive, KH, repreſenting the logarithm of the proper fraction HG, ſhall be negative, or in a contrary direction: and this is all that is meant when it is ſaid [art. 24.], that the logarithms of proper fractions are leſs than 0.

51. Hence numbers are not to one another accurately as their logarithms [art. 30.] for if they were, A, D, L, G, the extremities of the ordinates, would be in a ſtraight line [conv. 4. 6.]

52. Several properties of this curve have been diſcovered: ſuch as, that if two ſtraight lines AC, DF, be drawn, touching it in any two points A, D, meeting the axis BH in C, F; and if ordinates AB, DE, be drawn from the points of contact meeting the axis in B, E, the ſubtangents BC, EF, ſhall be equal; that the rectangle contained by the ſubtangent BC, and by the ordinate BA, is equal to the ſpace contained by the ordinate BA, and by the axis BH, and the curve ALG, continued *in infinitum*; that the curve itſelf is rectifiable, that is, that a ſtraight line can be found equal to it, &c. See *Hugen. Epiſt. de curvis quibuſdam peculiaribus.*

Method of conſtructing tables of logarithmic ſines, tangents, and ſecants.

53. The logarithmic ſines, tangents, and ſecants, are the logarithms of the natural ſines, tangents, and ſecants.

Example. To find the logarithmic ſine of 23°.

If the radius of the circle from which the table of natural ſines is conſtructed be 1000000000, which is neceſſary for greater accuracy, the natural ſine of 23° will be found to be 390731284. But becauſe tables of logarithms do not commonly extend to ſo great a number, cut off, toward the left hand, ſo many figures as the table extends to, ſuppoſe five, 39073: find the logarithm of this laſt number, which the table ſhews to be 4.5918768: add as many cyphers to 39073 as there are of figures in the natural ſine more than in it; that is, 3907300000, and unit to the index of its logarithm for every cypher that is added: then the logarithm of 3907300000 will be 9.5918768. Alſo the difference between the logarithm of 39073 and that of 39074, the next greater number in the table is 111; therefore, as 100000 to 111, the tabular difference, ſo is 11284, the difference between 390731284 and 3907300000 to a fourth proportional 12. If this fourth proportional 12 be added to the former logarithm 9.5918768, the ſum 9.5918780 ſhall nearly be the logarithmic ſine of 23°. The logarithmic ſine of any other number of degrees or minutes is found in the ſame manner.

54. When the logarithmic ſines are found to each degree and minute of the quadrant, it is eaſy to find the logarithmic tangents; for the coſine of any arc is to its ſine as radius to its tangent. If therefore the logarithm of the coſine be ſubtracted from the ſum of the logarithms of the ſine and radius, the remainder ſhall be the logarithm of the tangent of the ſame arc.

55. And becauſe the coſine of any arc, radius, and the ſecant of the ſame arc, are proportionals; if the logarithm of the coſine be ſubtracted from twice the logarithm of the radius, the remainder ſhall be the logarithm of the ſecant.

56. The logarithm of the radius is 10, with as many cyphers as there are decimal places in the table. The logarithm of the radius in Napier's firſt tables is 0: but he ſoon ſaw it would be more convenient to make it 10, as well as to alter the form of his logarithms from the hyperbolical to the common; both of which he directs to be done in the appendix to the ſecond edition, publiſhed after his death by his ſon.

SECT.

S E C T. III.

Conſtruction of the Plane Scale.

PROP. I. PROB. *Plate* 3. *fig.* 6.

1. *To divide a ſtraight line* AB *into any number of equal parts.*

From the point A draw AC of any length, containing any angle with AB, and from B draw BD [31. 1.] parallel to AC; from A towards C, and from B towards D, take off the number of equal parts into which AB is to be divided, each of them greater than each of the parts of AB; join the correſponding points of diviſion in AC, BD; the ſtraight line AB ſhall then [9. & 10. 6.] be divided into the number of equal parts required.

Otherwiſe. Fig. 7.

Draw the ſtraight line CD, terminated in C, but unlimited towards D, parallel to AB; join CA, and produce it towards E: from C towards D take off the number of equal parts required as in the former method; join DB meeting CA in E, and from

E

E draw ſtraight lines to each of the points of diviſion in CD. Theſe ſtraight lines ſhall [4. 6.] divide AB into the number of equal parts required.

2. *The ſcales of equal parts. Plate* 2. *fig.* 1.

The ſtraight line AB is divided into any number of equal parts [1. of this] in the points 0, 1, 2, 3, &c. each of the parts being ſome aliquot part or parts of an inch. A*o* is divided into ten equal parts: if therefore each of the large diviſions AB be reckoned ten, each of the diviſions in A*o* will be unit; and if each of the large diviſions in AB be reckoned a hundred, each of the diviſions in A*o* will be ten, &c.

3. To take off from this ſcale a line of a given number of equal parts, for example 65. Extend the compaſſes from 6 on the large, to 5 on the ſmall diviſions, it is the line required. In like manner, the diſtance from 7 on the large, to 2 on the ſmall diviſions, is a line of 72 equal parts, &c.

4. This ſcale is rendered more uſeful by drawing CD parallel to AB, and dividing C*o*, which is equal to A*o*, into twelve equal

qual parts. By this means, if the large diviſions in CD be reckoned feet, the diviſions in C*o* will be inches, &c. Thus the diſtance from 7 in the large diviſions upon *o*D, to 8 on the ſmall diviſions from *o* to C, expreſſes a line of 7 feet 8 inches; and ſo on.

5. In order to enlarge or diminiſh a given plane, it is convenient to have a variety of theſe ſcales as in the figure. Numbers are ſet down towards A, ſhewing into how many equal parts the inch is divided in each ſcale.

The diagonal ſcale of equal parts. Fig. 2.

6. A right-angled parallelogram ABCD is deſcribed. One of its longer ſides BC is divided into any number of equal parts in the points E, 1, 2, 3, 4, &c. and through each of theſe points ſtraight lines are drawn parallel to BA, meeting AD. BE and AF are each of them [1. of this] divided into ten equal parts; through the firſt point of diviſion from F towards A, a ſtraight line is drawn to E; and through every other point of diviſion from F to A incluſively, ſtraight lines are drawn parallel to

the firſt, meeting EB in 1, 2, 3, 4, &c. Laſtly, BA is divided into ten equal parts in the points 1, 2, 3, 4, &c. and through each point of diviſion a ſtraight line is drawn parallel to BC.

7. The line BE is now divided into an hundred equal parts. To take off any number of theſe equal parts, for example 64; obſerve the point where the line that proceeds from 6 upon BE, cuts the line that proceeds from 4 upon BA; the diſtance between this point and EF contains 64 of the equal parts, whereof BE contains an hundred. In like manner, the diſtance from the point where the line that proceeds from 8 upon BE, cuts the line that proceeds from 5 upon BA, to EF, will contain 85 equal parts of the hundred. To take off from this ſcale a line of 365 equal parts, obſerve the point where the line that proceeds from 6 upon BE, cuts the line that proceeds from 5 upon BA; the diſtance from this point, to the line that proceeds from 3 upon EC, is a line of 365 equal parts; and ſo on.

Another diagonal ſcale, of one half the length of the firſt, is commonly put on the other end at D and C; its large diviſions are marked from D towards A.

8. To deſcribe the plane of a given triangle ABC, by the ſcale of equal parts. Plate 3. fig. 8.

Take off the ſcale a line of the ſame number of equal parts with BC; let this line be EF; again, take off the ſame ſcale a line of the ſame number of equal parts with BA, and with that diſtance from the centre E deſcribe an arc: laſtly, from the centre F, with a diſtance equal to the number of equal parts in CA, deſcribe another arc, cutting the former in D, and join ED and FD; the triangle DEF is [22. & 23. 1.] the plane of the triangle ABC.

9. If, in the ſame manner, triangles DKE, DLF, be deſcribed upon each of the ſtraight lines ED, FD, the ſides of which contain the ſame number of equal parts with the ſides of the triangles AGB, AHC, each of each; the five-ſided rectilineal figure DKEFL ſhall be [20. 6.] the plane of the five-ſided rectilineal figure AGBCH. In like manner, the plane of any given rectilineal figure may be deſcribed.

The protracting ſcales. Plate 2. fig. 3.

10. Line of chords. A circle ADBE is deſcribed from the centre C; the quadrant DA is [30. 3. & cor. 15. 4.] divided into ninety degrees *, and DA is joined; from the centre D, with the diſtance of each point of diviſion in the quadrant, arcs are deſcribed meeting DA in 10, 20, 30, &c.; D10, D20, D30, &c. are the chords of 10, 20, 30, &c. degrees. This line with its diviſions is transferred to the ſcale, and marked CH.

11. To make at the point A, plate 3. fig. 9. in the ſtraight line AB, an angle of any number of degrees; for example forty, by the line of chords.

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* In order to divide the circumference of a circle into degrees and minutes, it is requiſite to divide a given circumference into three equal parts. This is the famous problem of the triſection of an angle, of which ſeveral ſolutions have been given by ancient and modern geometricians. No general ſolution of it, ſo far as I know, is, or can be given, but mechanically, or by means of the higher geometry. See 7. & 8. *Archim. de lin. ſpiral.*; *Vietæ Theor. ad angulares ſectiones*; *Newt. Arith. Univer. Appen. de æqua. conſtr. lineari.*; 1. & 2. *Appen. Simſ. Con. Ed. 2.*

Take the chord of ſixty degrees from the ſcale; from the centre A, with that diſtance, deſcribe an arc CD; and from the centre C, with the diſtance equal to the chord of forty degrees, deſcribe another arc cutting the former in D; join AD; DAC is an angle of forty degrees.

12. To find of how many degrees any angle EAB conſiſts, by the line of chords.

From the centre, with the diſtance equal to the chord of ſixty degrees, deſcribe an arc meeting AB, AE, in C, D; the diſtance CD, applied to the line of chords from the beginning of it, gives the number of degrees of the angle EAB.

13. Line of ſines. Plate 2. fig. 3. CD is drawn, and through each point of diviſion in the quadrant DA, a perpendicular is drawn to CD; theſe perpendiculars, which [def. 2. ſect. 1.] are the ſines of the arcs D10, D20, D30, &c. are transferred to the ſcale, and marked *Si*. The natural ſine of an angle is found by this line.

14. Line of tangents. From the point D, DG is drawn at right angles to DC; and from the centre C, ſtraight lines are drawn through each point of diviſion in the quadrant DA, meeting DG in 10, 20, 30, &c.

The ſtraight lines D10, D20, D30, &c. which [def. 3. ſect. 1.] are the tangents of the arcs D10, D20, D30, are transferred to the ſcale, and marked T*a*. By means of it, the natural tangent of a given angle is found.

15. Line of ſecants. The ſtraight lines intercepted between the centre and the extremities of the tangents, transferred to the ſcale, make [def. 4. ſect. 1.] a line of ſecants. It is commonly made the continuation of the line of ſines upon the ſcale.

16. Line of half tangents. If CA be joined, and ſtraight lines drawn from E to each point of diviſion in the quadrant AD, cutting CA in 10, 20, 30, &c. CA ſhall be divided into a line of half tangents. Becauſe [20. 3.] the angle at the centre is double of the angle at the circumference, ſo the lines C10, C20, C30, &c. on CA, are the tangents of 5, 10, 15, degrees of a circle of this radius. This line on the ſcale is marked S. T.

17. Line of rumbs. A quadrant BE is divided into eight equal parts; each of theſe equal parts is divided into four equal parts; the diſtance from B to each point of divi-

ſion in the quadrant being ſet off on BE, divide it into a line of rumbs.

Of the Logarithmic ſcales.

18. Line of numbers. This line, marked *Num.* on the ſcale, is conſtructed by means of a ſcale of equal parts of the ſame length with the intended line of numbers, and a table of logarithms. The ſcale of equal parts conſiſts of twenty equal primary diviſions, marked 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; each of theſe diviſions is divided into ten equal diviſions more; and each of theſe again into as many as their lengths will admit of: therefore, if the primary diviſions be reckoned hundreds, the next ſubdiviſions will be tens, and the next to theſe units. The line of numbers is biſected in 1. If the beginning of the line of numbers be reckoned unit, the middle point of it is to be reckoned ten, and the other end of it an hundred. In like manner, if the beginning be reckoned ten, the middle point is an hundred, and the other end a thouſand; or if the beginning be reckoned one tenth, the middle point is unit, and the other end ten.

19. The logarithmic diſtances from unit to ten, and from ten to an hundred, are taken from the ſcale of equal parts in this manner. Obſerve, in a table of logarithms, the three firſt figures of the decimal part of the logarithm of 2, viz. 301; then take 301 off the ſcale of equal parts, that is, three of the primary parts, together with one hundredth part of the next primary part, and ſet it off from the beginning of the line of numbers to 2; the diſtance from the beginning of the line of numbers to 2 is the logarithmic diſtance of 2. Next obſerve the three firſt figures of the decimal part of the logarithm of 3, viz. 477; take 477 from the ſcale of equal parts, as before, and ſet it off from the beginning of the line of numbers to 3. In like manner, the number of equal parts anſwering to the three firſt figures of the decimal part of the logarithm of 4 is to be ſet off from the beginning of the line of numbers to 4; and ſo on to ten. The logarithmic diſtances from 1 to 2, from 2 to 3, from 3 to 4, &c. found as above, are to be ſet off on the other half of the line of numbers, from 1 to 2, from 2 to 3, &c. Hence, if the figures marked upon the firſt half of the ſcale be reckoned units, thoſe

on the other half ſhall be tens, viz. 20, 30, 40, &c. to an hundred; and if the figures marked on the firſt half be reckoned tens, thoſe on the other half will be hundreds, &c.

20. In order to find the logarithmic diſtances from ten to twenty, from twenty to thirty, &c. Obſerve, in the table of logarithms, the three firſt figures of the decimal part of the logarithm of 11, viz. 041; take 041 from the ſcale of equal parts, that is, four of the ſecondary parts, and one tenth of another; this diſtance, ſet off from 1 towards 2, both from the beginning and from the middle of the line of numbers, gives the point expreſſing 11. In like manner, the number of equal parts, anſwering to the three firſt figures of the decimal part of the logarithm of each of the numbers 12, 13, 14, &c. to 20, being ſet off from the beginning and middle of the line of numbers, gives the point expreſſing 12, 13, 14, &c. and if theſe points from 1 to 2 at the beginning of the line be reckoned tenths, (as when the beginning is reckoned unit), thoſe from 1 to 2 at the middle ſhall be units; or if theſe from the beginning to 2 be reckoned units, (as when the beginning

ning is reckoned ten), thoſe from the middle to 2 ſhall be tens, &c. In like manner, the points expreſſing 21, 22, 23, &c. 31, 32, 33, &c. are to be found and marked from 2 towards 3, and from 3 towards 4, on each half of the line; and ſo on.

The lines marking theſe points laſt found, from the middle to 2, from 2 to 3, from 3 to 4, from 4 to 5, are made long, in order to diſtinguiſh them from the following ſubdiviſions.

21. If the equal parts anſwering to the firſt figures of the logarithms of any number of the fractions between ten and eleven, and between eleven and twelve, &c. be ſet off from 1 at the middle of the line towards the point anſwering to eleven, and from eleven towards twelve, &c. the line of numbers ſhall be ſo divided as to anſwer a number of three or more places of figures.

22. Multiplication by the line of numbers. Extend the compaſſes from the beginning of the line to one of the factors; the ſame diſtance ſhall reach from the other factor to the product. *Example.* To multiply 4 by 9. Extend the compaſſes from 1, at the beginning of the line, to 4; the ſame diſtance ſhall reach from 9 to the product 36.

23.

23. Diviſion by the line of numbers. Extend the compaſſes from the dividend to the diviſor; the ſame diſtance ſhall reach from unit to the quotient. Thus, the diſtance from 72 to 9 ſhall reach from unit to 8, which is the quotient of 72 divided by 9.

24. Proportion by the line of numbers. Extend the compaſſes from the firſt term of the proportionals to the ſecond; the ſame diſtance, taken in the ſame direction, ſhall reach from the third term of the proportionals to the fourth: ſo the diſtance from 3 to 6 ſhall reach from 4 to 8; and ſo on.

25. The line of logarithmic ſines. This line, marked *Sin.* may be conſtructed in this manner. Obſerve, in a table of logarithmic ſines, the three firſt figures of the decimal part of the logarithmic ſine of ten degrees, viz. 239; take 239 from the ſcale of equal parts, (as was done in the line of numbers), and ſet it off from the latter end of the ſcale (marked 90) towards the beginning, marking the point where it falls 80. In like manner, obſerve the three firſt figures of the decimal part of the logarithmic ſine of twenty degrees, viz. 534; take 534 from the ſcale of equal parts, and

ſet it off from 90 as before, marking the point it reaches 70. The other points of diviſion, anſwering to any degrees and minutes, are found in the ſame manner.

26. The line of logarithmic tangents. This line is conſtructed in the ſame manner as the line of logarithmic ſines: only the logarithms of the tangents are to be taken out of the table. The left-hand end of it is marked 45; becauſe the tangent of forty-five degrees is equal to radius. The tangents of degrees, above forty-five, are to be counted backwards: ſo the tangent of forty degrees is alſo the tangent of fifty, the tangent of thirty degrees is alſo the tangent of ſixty, as is commonly marked on ſcales. The chief uſes of the logarithmic ſcales of ſines and tangents ſhall be mentioned afterwards in ſolving the caſes of plane and ſpherical triangles.

27. The lines of logarithmic ſines of rumbs, and tangents of rumbs, (marked S.R. T.R.), are conſtructed as the lines of logarithmic ſines and tangents; only the logarithms of the ſines and tangents of the degrees that correſpond to each point of the compaſs muſt be taken out of the table.

28. The line of logarithmic verſed ſines (marked V. S.) may be conſtructed, by ſubtracting the logarithmic coſines of ten, twenty, thirty, &c. degrees from the logarithm of the radius, and ſetting off the equal parts anſwering to the firſt figures of the decimal part of the remainder, as on the other ſcales.

29. *Of the Sector.* *Plate* 2. *fig.* 4.

The lines of lines, marked *Lin.* is divided into ten equal parts; each tenth part of it is divided into ten equal parts more, and each of theſe again into halves, or more equal parts, if their lengths admit of it.

30. The lines of ſines marked *Sin.* the lines of chords marked *Cho.* and the lines of tangents marked *Tan.* are each of them equal to the lines of lines; and ſo adjuſted, that the angle contained by the lines of lines at the centre of the ſector is equal to that contained by the lines of ſines, by the lines of tangents, and by the lines of chords.

31. To conſtruct the lines of ſines. Obſerve, in a table of natural ſines, the three or four firſt figures of the natural ſine of ten degrees, viz. 173; take 173 off the

line of lines; from the centre, with this diſtance, mark the lines of ſines: theſe points give the ſine of ten degrees. In like manner, 342, viz. the three firſt figures of the natural ſine of twenty degrees being taken from the line of lines, and ſet off from the centre on the lines of ſines, gives the ſine of twenty degrees; and ſo on to ninety degrees. In the ſame manner, the points anſwering to ſingle degrees, and to as many minutes as the length of the ſector admits of, are to be found.

32. The lines of chords. The ſeveral points in theſe lines may be found by taking from the table of natural ſines the ſine of half the propoſed number of degrees or minutes, and ſetting off twice this diſtance from the centre of the ſector upon the lines of chords. Thus, to find the points anſwering to the chord of ten degrees; obſerve, in the table, the two firſt figures of the natural ſine of five degrees, viz. 87; take the double of 87 (174) from the line of lines, and ſet it off from the centre upon the lines of chords; the points ſo found anſwer to the chord of ten degrees; and ſo on to ſixty degrees.

33. Lines of tangents. Theſe lines are conſtructed as the lines of ſines; only the firſt figures of the natural tangents are to be taken from the table.

The lines of tangents above forty-five degrees (marked *tan.*) begin one fourth part of the radius of the other tangents from the centre, and are conſtructed by taking from the line of lines the fourth part of the natural tangent of the given degree, and ſetting it off from the centre. Thus, the four firſt figures of the natural tangent of ſixty degrees are 1732; the fourth part of which is 433: if 433 be taken from the line of lines, and ſet off from the centre of the ſector, it ſhall give the tangent of ſixty degrees, &c.

34. The lines of ſecants are conſtructed and placed as thoſe of tangents above forty-five degrees, by taking the fourth part of each natural ſecant from the table, and from the line of lines.

35. The lines of polygons (marked *Pol.*) are conſtructed by taking the diviſions of it from a ſcale of chords, in which the chord of ninety degrees is equal to the radius of the ſector. Thus the chord of thirty degrees being taken from ſuch a ſcale of

chords, and ſet off from the centre of the ſector on theſe lines, gives the points 12; the chord of thirty-ſix degrees being ſet off in the ſame manner gives the points 10, &c. If the radius of a circle be extended from 6 to 6 on theſe lines, the diſtance from 4 to 4 ſhall be the ſide of a ſquare, the diſtance from 5 to 5 ſhall be the ſide of a pentagon inſcribed in that circle, and ſo on.

36. The lines of numbers, logarithmic ſines, tangents, &c. are commonly put upon the ſector, ſo as to be uſed when it is fully open. Some ſectors have alſo the lines of the protracting ſcales, &c.

37. The diſtance from the centre of the ſector on any of the lines that proceed from the centre is called *lateral diſtance*; and the diſtance from any point in a line on one ſide of the ſector to the correſponding point on the other ſide is called *parallel diſtance*, as from 40 to 40 on the lines of lines.

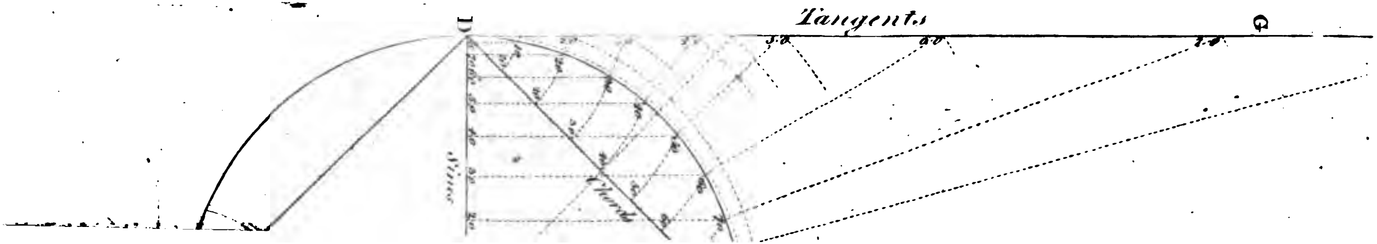
38. Multiplication by the lines of lines. Make the lateral diſtance of one of the factors the parallel diſtance of 10; the parallel diſtance of the other factor is the product [4. 6.]. *Example.* To multiply 5 by 6. Extend the compaſſes from the centre of the ſector to 5 on the primary diviſions, and

open the ſector till this diſtance become the parallel diſtance from 10 to 10 on the ſame diviſions; then the parallel diſtance from 6 to 6, extended from the centre of the ſector, ſhall reach to 3, which is now to be reckoned 30. At the ſame opening of the ſector the parallel diſtance of 7 ſhall reach from the centre to 35, that of 8 ſhall reach from the centre to 40, &c.

39. Diviſion by the lines of lines. Make the lateral diſtance of the dividend the parallel diſtance of the diviſor, the parallel diſtance of 10 is the quotient. Thus, to divide 30 by 5, make the lateral diſtance of 30, viz. 3 on the primary diviſions, the parallel diſtance of 5 of the ſame diviſions; then the parallel diſtance of 10, extended from the centre, ſhall reach to 6.

40. Proportion by the lines of lines. Make the lateral diſtance of the ſecond term the parallel diſtance of the firſt term, the parallel diſtance of the third term is the fourth proportional. *Example.* To find a fourth proportional to 8, 4, and 6, take the lateral diſtance of 4, and make it the parallel diſtance of 8; then the parallel diſtance of 6, extended from the centre, ſhall reach to the fourth proportional 3.

In the ſame manner, a third proportional is found to three numbers. Thus, to find a third proportional to 8 and 4, the ſector remaining as in the former example, the parallel diſtance of 4, extended from the centre, ſhall reach to the third proportional 2. In all theſe caſes, if the number to be made a parallel diſtance be too great for the ſector, ſome aliquot part of it is to be taken, and the anſwer is to be multiplied by the number by which the firſt number was divided. Thus, if it were required to find a fourth proportional to 4, 8, and 6; becauſe the lateral diſtance of the ſecond term 8 cannot be made the parallel diſtance of the firſt term 4, take the lateral diſtance of 4, viz. the half of 8, and make it the parallel diſtance of the firſt term 4; then the parallel diſtance of the third term 6 ſhall reach from the centre to 6, viz. the half of 12. Any other aliquot part of a number may be uſed in the ſame way. In like manner, if the number propoſed be too ſmall to be made the parallel diſtance, it may be multiplied by ſome number, and the anſwer is to be divided by the ſame number.



Tangents
G
D
Chords
Sines
10
20
30
40
50
60
70

41. To protract angles by the lines of chords, plate 3. fig. 9. *Example.* At the point A, in the ſtraight line AB, to make an angle of any given number of degrees, ſuppoſe forty. Extend the compaſſes from the centre of the ſector to 60 on the line of chords; from the centre A, with that diſtance on AB, deſcribe an arc CD; open the ſector till the lateral diſtance of 60 on the line of chords become the parallel diſtance from 60 to 60 on the ſame; the ſector remaining ſo, take the parallel diſtance from 40 to 40 on the lines of chords; and from C mark the arc CD with it in D. If DA be joined, DAB ſhall be an angle of forty degrees. In like manner, if it be required to find the number of degrees contained in an angle EAB, the arc CD is to be deſcribed, and the ſector opened as before; then the diſtance from C to D ſhall reach from 40 to 40 on the lines of chords.

42. To make a rectilineal figure, fig. 8. ſimilar to the given rectilineal figure AGBCH by means of the lines of chords on the ſector, and a line of equal parts. Make EF of the ſame number of equal parts with BC; make the angles EFL, FEK, equal to the angles BCH, CBG, each to each, (by the preceding),

ding), and make FL, EK, of the ſame number of equal parts with CH, BG, each to each, and ſo on. The figure DKEFL is ſimilar to the figure AGBCH.

N. B. When the angle to be made by the lines of chords at any point F contains more than ſixty degrees, it muſt be divided into two parts.

43. To inſcribe, in a given circle, a regular polygon by means of the lines of polygons. Make the radius of the circle the parallel diſtance from 6 to 6 on the lines of polygons; the parallel diſtance of 4, 5, 6, 7, 8, &c. ſhall each of them be the ſide of a ſquare, the ſide of a pentagon, the ſide of a hexagon, &c. inſcribed in that circle. It is manifeſt the ſame thing may be done by means of the lines of chords. The chief uſes of the other lines upon the ſector will be deſcribed in the ſections which treat of plane and ſpherical trigonometry, and the projection of the ſphere.

SECT. IV.

Plane Trigonometry.

PROP. I. THEOR. *Plate* 3. *fig.* 10.

1. *If* BA, *the hypothenuſe, that is, the ſide oppoſite to the right angle of any right-angled triangle* ABC, *be made radius, each of the other ſides ſhall be the ſine of the angle oppoſite to itſelf.*

From the centre B, with the diſtance BA, deſcribe the circle ADE, meeting BC produced, in D: it is manifeſt [def. 2.], that AC is the ſine of the arc AD, that is, of the angle ABC. For the ſame reaſon, if from the centre A, with the ſame diſtance AB, a circle be deſcribed, BC ſhall be the ſine of the angle BAC.

Cor. 1. BA and AC, fig. 11. the hypothenuſe, and one ſide of a right-angled triangle ABC, being each of them given, to find the angle ABC oppoſite to AC; for example, let BA be 406 equal parts, and AC 220.

(1) Solution by the table of natural ſines. A fourth proportional to the three terms

406,

406, 220, and the radius 10000000, viz. 5417082 is the natural ſine of the angle ABC, thirty-two degrees forty-eight minutes.

(2) Solution by the logarithmic tables of numbers and ſines. Add 2.3424227, which is the logarithm of AC 220, to that of radius 10.0000000; and from the ſum 12.3424227 ſubtract 2.6085260, which is the logarithm of BA 406, the remainder 9.7338967 is the logarithmic ſine of the angle ABC, thirty-two degrees and forty-eight minutes, as before.

(3) Solution by the logarithmic ſcales. Extend the compaſſes on the line of numbers from 406 to 220, the ſame diſtance ſhall reach on the line of ſines from 90 to the number of degrees and parts of a degree ſought.

(4) Solution by the ſector. Take 406 in the compaſſes from one of the lines of lines; open the ſector till this diſtance become the parallel diſtance from 90 to 90 on the lines of ſines; then take 220 from the line of lines: this laſt diſtance being made a parallel diſtance on the lines of ſines, ſhall give the number of degrees and minutes ſought, in this caſe thirty-two degrees and

forty-eight minutes, the ſame as by the former methods.

Cor. 2. The hypothenuſe BA, and an angle ABC, being given, to find the oppoſite ſide AC. AC is a third proportional to the three terms, radius, the ſine of the angle ABC, and the hypothenuſe BA, which is found by any of the preceding methods.

Cor. 3. The angle ABC, and a ſide AC being given, to find the hypothenuſe BA. BA is a third proportional to the three terms, viz. the ſine of the angle ABC, radius, and AC.

PROP. II. THEOR. *Fig.* 10.

2. *If* BA, *the hypothenuſe of a right-angled triangle* ABC, *be made radius, each of the other ſides ſhall be the coſine of the angle adjacent to itſelf.*

Becauſe BCA is a right angle, the other angles CBA and CAB are together [32. 1.] equal to a right angle; therefore any one of theſe angles [def. 5.] is the complement of the other; wherefore (by the preceding and def. 6.) BC and CA are each of them the coſines of each of the adjacent angles CBA, CAB. Q. E. D.

Cor. Hence, the coſine of the angle ABC, fig. 11. is a third proportional to the three terms BA, BC, and radius.

Example. Let BA be 406, and BC 334, to find the angle ABC.

(1) By the logarithmic tables. Add the logarithm of BC 334, to the logarithmic radius, and from the ſum ſubtract the logarithm of BA 406, the remainder ſhall be the logarithmic coſine of the angle ABC.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of numbers from 406 to 334; the ſame diſtance ſhall reach on the line of ſines, from 90 to the number of degrees which is the complement of the angle ABC.

(3) By the ſector. Open the ſector till the lateral diſtance of 406 from one of the lines of lines become the parallel diſtance of 90 and 90 on the lines of ſines; then the lateral diſtance of 334 from a line of lines, applied parallel wiſe on the lines of ſines, ſhall fall upon the number of degrees which is the complement of the angle ABC. In like manner, if the hypothenuſe AB, and the angle ABC, be given, the ſide BC is found, becauſe it is a fourth proportional to the three terms, viz. radius, the coſine of

the angle ABC, and the hypothenuſe BA. Thus, if the number of equal parts in BA be taken from one of the lines of lines upon the ſector, and made a parallel diſtance from 90 to 90 upon the lines of ſines, then the parallel diſtance of the number of degrees which is the complement of the angle ABC ſhall reach on one of the lines of lines from the centre of the ſector to the number of equal parts in BC. Alſo, if the angle ABC and the ſide BC be given, the hypothenuſe BA is found; becauſe it is a fourth proportional to the three terms, the coſine of the angle ABC, radius, and BC.

PROP. III. THEOR. *Fig.* 10.

3. *If* BC, *one of the ſides containing the right angle of a right-angled triangle* ABC, *be made radius, the other ſide* CA *ſhall be the tangent of the oppoſite angle* ABC, *and the hypothenuſe* BA *ſhall be the ſecant of the ſame angle.*

From the centre B, with the diſtance BC, deſcribe the circle CFE, meeting BA in F. It is manifeſt [def. 3.], that CA is the tangent of the arc CF, or of the angle ABC; and [def. 4.] BA is the ſecant of the ſame arc CF, or of the angle ABC. For the ſame reaſon, if from the centre A, with the diſtance AC, a circle be deſcribed, CB ſhall be the tangent of the angle BAC, and AB ſhall be the ſecant of the ſame angle. Q.E. D.

Cor. 1. Hence, if a ſide BC, fig. 11. and the adjacent acute angle ABC of a right-angled triangle ABC, be given, the other ſide CA is found; becauſe it is a third proportional to the three terms, viz. radius, the tangent of the angle ABC, and the ſide BC.

Thus, if BC be 334 equal parts, and the angle ABC thirty-two degrees and forty-eight minutes, to find CA.

(1) By the logarithmic tables. Add the logarithm of BC 334 to the logarithmic tangent of the angle ABC 32° 48′, and from the ſum ſubtract the logarithm of the radius, the remainder ſhall be the logarithm of CA 220.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of tangents from radius, viz. 45° to 32° 48′; the ſame diſtance ſhall reach on the line of numbers from 334 to 220.

(3) By the ſector. Take 334 in the compaſſes from one of the lines of lines; make that a parallel diſtance from 45 to 45

upon the lines of tangents; then take the parallel diſtance of 32° 48′ on the lines of tangents, and it ſhall reach from the centre of the ſector upon a line of lines to 220, as before.

Cor. 2. If BC and CA be each of them given, the angle ABC is found; becauſe its tangent is a fourth proportional to the three terms, BC, CA, and radius. In like manner, the angle BAC is found *mutatis mutandis.*

Cor. 3. If CA, and the oppoſite angle ABC, be given, the adjacent ſide BC is found; becauſe it is a fourth proportional to the three terms, viz. the tangent of the angle ABC, radius, and the other ſide AC.

PROP. IV. THEOR. *Fig.* 12.

4. *Any two ſides* AB, BC, *of any triangle* ACB, *are to one another as the ſines of the angles* BCA, BAC, *oppoſite to theſe ſides; that is,* AB *is to* BC, *as the ſine of the angle* BCA, *to the ſine of the angle* BAC.

At the point A, in the ſtraight line AC, make [23. 1.] the angle CAF equal to the angle ACB; from the centre A, with the diſtance AB, deſcribe the circle BGF, meeting AC and AF in G and F; from F and B draw [12. 1.] FE and BD at right angles to AC. Becauſe the angle AEF is equal to the angle CDB, each being a right angle, and that the angle EAF is equal to the angle DCB by conſtruction, the triangles FAE, BCD, are [32. 1.] equiangular; wherefore [4. 6.] AF is to FE, as CB to BD; alternately, AF or AB is to CB, as FE to BD. But FE [def. 2.] is the ſine of the arc FBG, or of the angle FAC, which, by conſtruction, is equal to the angle ACB, and BD is the ſine of the arc BG, or of the angle BAC; therefore AB is to BC, as the ſine of the angle BCA to the ſine of the angle BAC. Q. E. D.

Cor. 1. Hence, if BA and BC, fig. 13. two ſides of any triangle BAC, be each of them given, and an angle BCA oppoſite to one of them, the angle BAC, oppoſite to the other ſide, is found; becauſe its ſine is a third proportional to the three terms BA, BC, and the ſine of the angle BCA.

Thus, let BA be 232 equal parts, BC 297, and the angle BCA forty-eight degrees, to find the angle BAC. (1) By the logarithmic tables. Add the logarithm of BC 297, to the logarithmic ſine of the angle BCA 48°, and from the ſum ſubtract the logarithm of BA 232, the remainder ſhall be the logarithmic ſine of the angle BAC.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of numbers from 232 to 297; the ſame diſtance ſhall reach on the line of ſines from 48° to the number of degrees of the angle BAC.

(3) By the ſector. Make BA 232, viz. the ſide oppoſite to the given angle, the parallel diſtance of 48° on the lines of ſines; then the lateral diſtance of the other ſide BC 297, applied parallel wiſe on the lines of ſines, ſhall fall on the number of degrees of the angle BAC.

Cor. 2. If two angles BCA, BAC, be each of them given, and a ſide BA oppoſite to one of them, the ſide BC, oppoſite to the other angle, is found; becauſe it is a fourth proportional to the three terms, the ſine of the angle BCA, the ſine of the angle BAC, and the ſide BA. In like manner, if the ſide BC, and each of the angles BAC, BCA, be given, the ſide BA is found.

N. B. When two ſides, and an angle oppoſite to one of them, are given, in order to find the angle oppoſite to the other ſide, the ſide oppoſite to the given angle is to be made the firſt term of the proportionals.

And when two angles, and a ſide oppoſite to one of them, are given, in order to find the ſide oppoſite to the other angle, the ſine of the angle oppoſite to the given ſide is to be made the firſt term of the proportionals.

It is to be obſerved likewiſe, that the ſolutions by this propoſition are ſometimes ambiguous; becauſe the ſine of an angle leſs than ninety degrees, is equal to the ſine of an angle as much greater than ninety degrees. The ambiguity is avoided, if the ſum of the other two angles be known; becauſe [32. 1.] the three angles of a triangle are together equal to two right angles.

PROP. V. THEOR. *Fig.* 14.

5. *The ſum of any two ſides,* BA *and* AC, *of any triangle* ABC, *is to the difference of the ſame ſides, as the tangent of half the ſum of the angles* ACB, ABC, *oppoſite to theſe ſides, to the tangent of half the difference of the ſame angles.*

Let AC be the greater ſide, ABC [18. 1.] is the greater angle. Produce BA, and make AE equal to AB, and AD equal to AC; make AG equal to AB or AE; BD is the ſum of the two ſides, BA and AC; ED or GC is their difference: join DC, EG, and they ſhall [2. 6.] be parallel; join BG, meeting DC in H; and from the centre B, with the diſtance BH, deſcribe the circle HKL, meeting BC in K, and BD in L: laſtly, biſect the angle EAG by the ſtraight line AF, meeting EG in F; the angle EAF is [32. 1.] equal to half the ſum of the angles ABC, ACB. Becauſe EA is equal to AG, and the angle EAF to the angle GAF, EG [4. 1.] is biſected at right angles by AF; but EA is equal to AB, therefore AF and BG [2. 6.] are parallel, and the angle EBG is equal to the angle EAF; that is, to half the ſum of the angles ABC, ACB; wherefore HBC is half their difference, and the angle BGE, or BHD, a right angle. And, becauſe each of the angles GHC, BHD, is a right angle, and that the angle GCH is equal [5. 1.] to the angle BDH, the triangles CGH, DBH, are equiangular; therefore [4. 6.] BD is to DH, as GC to CH; alternately, BD is to GC or ED, as DH to CH: But DH [def. 3.] is the tangent of the arc LH, or of the angle DBH, which is half the ſum of the angles ABC, ACB;

and CH is the tangent of the arc KH, or of the angle CBH, which is half their difference. Q. E. D.

Cor. 1. Hence, if each of two ſides, BA, AC, fig. 15. of any triangle ABC, and the angle BAC which theſe ſides contain, be given, each of the angles ABC, ACB, is found. Thus, let BA be 287, AC 209 equal parts, and the angle BAC eighty-three degrees, the ſum of BA and AC is 496, their difference is 78: and becauſe BAC is an angle of 83 degrees, the ſum of the other two angles ABC, ACB, [32. 1.] is 97 degrees; the half of which is 48 degrees 30 minutes: to the three terms 496, 78, and the tangent of 48° 30′, find a fourth proportional; this fourth proportional ſhall be the tangent of half the difference of the angles ABC, ACB.

The ſolution.——(1) By the logarithmic tables. Add 1.8920946, which is the logarithm of 78 to 10.0531916, which is the logarithmic tangent of 48° 30′, and from the ſum 11.9452862 ſubtract 2.6954817, which is the logarithm of 496, the remainder 9.2498045 is the logarithmic tangent of half the difference of the angles ABC, ACB, viz. 10 degrees, and between 4 and 5 minutes.

(2) By the logarithmic ſcales. It was obſerved [ſect. 3. art. 26.], that the tangents of degrees above forty-five, on the line of logarithmic tangents, are to be counted backwards. Then, either the angle given, and the angle ſought, are each of them greater than forty-five degrees; or one of them is greater than forty-five degrees, and the other not greater: for if each of them be leſs than forty-five degrees, there is no need for counting backwards; if each of them be greater than forty-five degrees, each of them is found on the degrees counted backwards. Thus, if one of them be fifty degrees, and the other eighty, their diſtance is the ſame with that between forty and ten degrees. The intermediate degrees are counted in the ſame manner. But if one of them be greater than forty-five degrees, and the other not greater, the one of them is found on the degrees counted backwards, and the other on thoſe counted forwards, by two operations, which will be beſt underſtood by an example, ſuch as the preſent, viz. to find half the difference of the angles ABC, ACB. Extend the compaſſes

on the line of numbers from 496 to 78; obſerve how far this diſtance reaches from 45° on the line of tangents, and take the diſtance from the point it reaches to the number of degrees in the given angle 48° 30′, (which is to be counted backwards), this diſtance ſhall reach from 45 degrees to the number of degrees ſought (on thoſe counted forwards); the ſame as by the tables.

(3) By the ſector. Take, for example, the tenth part of 496, viz. 49.6 from one of the lines of lines; make this a parallel diſtance from 48° 30′, to 48° 30′ on the lines of tangents above 45; then take the parallel diſtance from 45 to 45 on the ſame lines of tangents, and make it a parallel diſtance from 45 to 45 on the lines of tangents below 45; the ſector remaining ſo, the tenth part of 78, viz. 7.8 ſhall reach from 10° 4′ to 10° 4′ on the lines of tangents below 45. In caſes of this kind, it is neceſſary that the part taken of the given number be ſuch as can be made a parallel diſtance of the number of degrees in the given angle on the lines of tangents above 45, and then that the parallel diſtance of 45 on it can be made

the parallel diſtance of 45 on the lines of tangents below 45.

Having thus found half the difference of the angles ABC, ACB, viz. 10° 4′, add it to half their ſum 48° 30′, and the ſum 58° 34′ ſhall be the number of degrees of the angle ACB, which [18. 1.] is the greater of the two; wherefore ABC is an angle of 38° 26′.

Cor. 2. If each of the angles ABC, ACB, and the ſum of the ſides AB, AC, be given, each of theſe ſides is found; becauſe half their difference is a fourth proportional to the three terms, the tangent of half the ſum of the angles ABC, ACB, the tangent of half their difference, and the ſum of the two ſides AB, AC.

PROP. VI. THEOR. *Fig.* 16.

6. *If a perpendicular be drawn from* A, *the vertex of any triangle* ABC, *meeting the baſe in* D, *the baſe* BC *is to the ſum of the two ſides* BA *and* AC, *as the difference of* BA *and* AC *to the difference of the ſegments of the baſe* BD *and* DC.

Let AC be the greater ſide. Becauſe [47. 1.] the ſquare of AC is equal to the ſquares of AD and DC, and the ſquare of AB to the ſquares of AD and DB, the exceſs of the ſquare of AC, above the ſquare of AB, is equal to the exceſs of the ſquares of AD, DC, above the ſquares of AD, DB; take away the common ſquare of AD, and the exceſs of the ſquare of AC, above the ſquare of AB, ſhall be equal to the exceſs of the ſquare of DC above the ſquare of DB. But [5. 2.] the exceſs of the ſquare of AC above the ſquare of AB is equal to the rectangle contained by the ſum and difference of AC and AB; and the exceſs of the ſquare of DC above the ſquare of DB is equal to the rectangle contained by the ſum and difference of DC and DB; wherefore the rectangle contained by the ſum and difference of AC and AB is equal to the rectangle contained by the ſum and difference of DC and DB; therefore [14. 6.] BC is to BA and AC, as the difference of BA and AC to the difference of BD and DC. Q. E. D.

Cor. 1. The three ſides BC, BA, and AC, of any triangle being each of them given, and the difference of the ſegments of the baſe BD, DC, being found, viz. a fourth proportional to the three terms BC, the ſum of BA and AC, and the difference of BA and AC; each of the ſegments of the baſe BD, DC, is found, becauſe the greater ſegment DC is equal to the half of BC, together with the half of the difference of BD and DC, and that BD is the remaining part of BC.

Cor. 2. Hence, if the three ſides BC, BA, and AC, fig. 16. of any triangle ABC be each of them given, each of the angles is found. Let BA be 217, AC 322, and BC 338, equal parts, the ſum of BA and AC is 539; their difference is 105. To 338, 539, and 105, find a fourth proportional 167.44: to the half of 338, which is 169, add the half of 167.44, which is 83.72; the ſum 252.72 is the greater ſegment DC; wherefore DB is 85.28. To find the angles: the coſine of the angle ACB [cor. 2. of this] is a fourth proportional to the three terms CA, CD, and radius; and the coſine of the angle ABC is a fourth proportional to the three terms BA, BD, and radius; wherefore [32. 1.] the third angle BAC is given.

Of

Of the number of caſes in the ſolutions of right-angled plane triangles.

7. The parts of any right-angled triangle ABC, fig. 11 that are conſidered in trigonometry, are five in number, viz. three ſides, and two acute angles; for the right angle being a conſtant quantity, is always given. Of theſe five parts, two muſt be given, in order to find a third; for radius, which is a conſtant quantity, is always one term of the four proportionals. The number of times that three different parts of five can be combined is ten; and the number of times that each two of theſe three parts can be combined with the third is three. But the caſes of right-angled plane triangles do not amount to ſo many. Some of theſe combinations are unneceſſary, (for to combine the hypothenuſe with one ſide and the angle oppoſite to that ſide, is the ſame thing with combining it with another ſide and the angle oppoſite to it), and for others there are not direct propoſitions. The combinations ariſing from the three propoſitions that have been demonſtrated, relating to right-angled triangles, are,

(1) When the hypothenuſe AB is made radius [1. of this], it is combined with one of the acute angles ABC, and AC the ſide oppoſite to that angle. By the ſame propoſition, the hypothenuſe AB is combined with the other acute angle BAC, and the oppoſite ſide BC. It is unneceſſary to combine the hypothenuſe with the two angles; becauſe when one of the acute angles of a right-angled triangle is given [32. 1.], the other is alſo given; and there is no propoſition directly for the combination that is made of the hypothenuſe with the two ſides. In the combination of the hypothenuſe with one of the acute angles, and the oppoſite ſide, there are three caſes contained: for either the hypothenuſe and the angle are given in order to find the ſide, or the hypothenuſe and the ſide are each of them given, in order to find the angle; or, laſtly, the ſide and angle are given, in order to find the hypothenuſe.

(2) When the hypothenuſe AB is made radius [2. of this], it is combined with the complement of one of the acute angles ABC and BC, the ſide adjacent to that angle. This combination contains alſo three caſes: for either the hypothenuſe and angle

are given in order to find the ſide, or the hypothenuſe and the ſide are each of them given in order to find the angle; or, laſtly, the ſide and angle are given in order to find the hypothenuſe.

But theſe laſt caſes are commonly omitted, becauſe they may be reduced to the preceding ones, depending upon prop. 1. of this. They are mentioned here only for the ſake of variety.

(3) When one ſide BC is made radius [3. of this], it is combined with the other ſide CA, and the angle ABC oppoſite to CA. This combination contains alſo three caſes: for either BC and the angle ABC are given in order to find the other ſide CA, or BC and CA are each of them given in order to find the angle ABC; or, laſtly, the ſide CA, and the oppoſite angle ABC, are given in order to find the ſide BC: So that there are properly three caſes of a right-angled triangle when the hypothenuſe is made radius, and other three, when one of the ſides is made radius. Some of theſe caſes lead to others to which the propoſitions do not relate directly. For example, the hypothenuſe AB, and the ſide AC, may each of them be given in order to find the other

ſide BC. In this caſe, the angles muſt be firſt found; then BC is found by means of the hypothenuſe and an angle.

The ſix caſes of right-angled plane triangles are ſet down in the following table. The firſt column contains the caſes, the ſecond contains the parts that are given, the third contains the parts that are ſought, and the fourth contains the proportionals.

The caſes of right-angled plane triangles. Plate 3. fig. 11.

	Given.	Sought.	Proportionals.
1.	The hypoth. BA, and a ſide AC.	The oppoſite angle ABC.	BA is to AC, as radius to the ſine of ABC [cor. 1. 1. of this].
2.	The hypoth. BA, and an angle ABC.	The oppoſite ſide AC.	Radius is to the ſine of ABC, as BA to AC [cor. 2. 1. of this].
3.	An angle ABC, and the oppoſite ſide AC.	The hyp. BA.	The ſine of ABC is to radius, as AC to AB [cor. 3. 1. of this].
4.	A ſide BC, & the adjacent angle ABC.	The other ſide CA.	Radius is to the tangent of ABC, as BC to CA [cor. 1. 3. of this].
5.	Two ſides BC, CA.	An angle ABC.	BC is to CA, as radius to the tangent of ABC [cor. 2. 3. of this].
6.	A ſide CA, & the oppoſite angle ABC.	The other ſide BC.	The tangent of ABC is to radius, as AC to CB [cor. 3. 3. of this].

O Of

Of the number of caſes in the ſolutions of oblique-angled plane triangles.

8. The parts of an oblique-angled plane triangle that are conſidered in trigonometry, are ſix, viz. three ſides, and three angles. Three of theſe ſix parts muſt be given in order to find a fourth; and this is the chief difference between a right and an oblique-angled triangle in the ſolution. The number of different combinations that can be made of four parts of ſix is fifteen; and each of theſe combinations contains four combinations of three parts of four with the fourth: but the caſes of oblique-angled plane triangles do not amount to ſo many by far, for the ſame reaſons that were given concerning right-angled plane triangles. The general propoſitions that have been demonſtrated, viz. the fourth, fifth, and ſixth, of this ſection, apply to right-angled plane triangles, as well as to oblique. By the fourth propoſition, any two ſides BA, BC, fig. 13. are combined with the two oppoſite angles BCA, BAC; and this combination of four parts, contains four combinations of three parts of four

with the fourth; becauſe any part of the four may be the one that is ſought: but theſe four combinations are reduced to two; becauſe, in a general ſolution, combining two ſides BA, BC, with one of the angles BAC, is the ſame thing with combining the ſame ſides with the other angle BCA: and combining two angles BAC, BCA, with one ſide BA, is the ſame thing with combining the ſame angles with the other ſide BC. The fifth propoſition is chiefly in order to reduce the ſolution of a triangle to one of the caſes ariſing from the fourth propoſition; and the ſixth propoſition is in order to find the angles by means of the three ſides: but the ſides cannot be found by means of the angles alone; becauſe the three angles of any plane triangle are together [32. 1.] equal to the three angles of any other plane triangle. The general caſes, therefore, in the ſolutions of plane triangles, which are commonly applied to oblique, are properly four, viz. two ariſing from prop. 4. one from prop. 5. and one from prop. 6. Theſe general caſes are ſet down in the following table, as thoſe of right-angled plane triangles are in the preceding.

Solutions of the caſes of oblique-angled plane triangles. Plate 3. fig. 13.

	Given.	Sought.	Proportionals.
1.	BA, BC, and the angle BCA.	The angle BAC.	BA is to BC, as the ſine of BCA, to the ſine of BAC [cor. 1. 4. of this]. Hence [32. 1.] the third angle ABC is found.
2.	The angles BCA, BAC, and the ſide BA.	The ſide BC.	The ſine of BCA is to the ſine of BAC, as BA to BC [cor. 2. 4. of this].
3.	BA, BC, and the angle ABC.	Each of the angles BAC, BCA	The ſum of BA and BC is to their difference, as the tangent of half the ſum of BAC and BCA to the tangent of half their difference; and if half the difference of BAC, BCA, be added to half their ſum, the ſum ſhall be the greater angle, and the remainder of the ſum of the two angles BAC, BCA, ſhall be the leſſer angle [cor. 1. 5. of this].
4. *fig.* 16	BA, BC, and AC.	Each of the three angles.	BC is to the ſum of BA and AC, as the difference of BA, AC, to the difference of BD, DC. Hence BD, DC, are each of them found, and BA is to BD, as radius to the coſine of the angle ABD. Likewiſe CA is to CD, as radius to the coſine of the angle ACD; wherefore [32. 1.] the angle BAC is found [cor. 2. 6. of this].

PART SECOND.

SECT. I.

The Principles of Perſpective.

1. PERSPECTIVE is an art by which the repreſentations of objects are geometrically deſcribed upon a plane, according as the objects appear to an eye given in poſition.

2. The rays of light that are reflected from the ſeveral points of an object, and interſect one another in the eye, are called *viſual rays.* Thus, plate 4. fig. 1. if an eye be at O, and if the line AB be the object, the ſtraight lines AO, BO, are viſual rays. In like manner, when the lines AC, BC, are the objects, AO, CO, BO, CO, are viſual rays.

3. The angle AOB, contained by the viſual rays AO, BO, is called *the viſual angle of the object or line* AB. Likewiſe AOC, BOC, are the viſual angles of AC, BC.

The magnitude of an object, when length alone is conſidered, is, in perſpective, eſtimated by the viſual angle; and the viſual angle of any object AB increaſes or diminiſhes reciprocally as the diſtance from the eye O [21. 1. & 4. 6.]. Thus, if EF be equal and parallel to BA, but nearer the eye, its apparent magnitude is greater than that of BA, in the ratio of the angle EOF to the angle BOA: for if EF meet OA in G, EG ſubtends the ſame viſual angle with BA; therefore the apparent magnitude of an object is reciprocally proportional to its diſtance from the eye.

4. The plane that paſſes through the viſual rays AO, BO, may be called the *viſual plane of the object or line* AB, which [2. 11.] is in the ſame plane; and the planes in which are the triangles ACO, BCO, are the viſual planes of AC, BC.

If the object viewed by the eye O, be any rectilineal figure ABC, the ſolid figure contained by the viſual planes, viz. AOC, COB, AOB, and by the object ABC, ſhall be [def. 12. 11.] a pyramid whoſe vertex is O, and whoſe baſe is the object ABC; and this pyramid is called the *optical*, or *viſual pyramid of the object* ABC. If the object be a circle, the rays that proceed from each point of the circumference to the eye are in the

the ſuperficies of a cone, whoſe vertex is in the eye, and whoſe baſe is the circle, which is the object.

6. If a plane DABE given in poſition, fig. 3. be erected between the eye O, and the point to be put in perſpective H, and if the point G, where the viſual ray HO meets the plane DABE, be found, this point ſhall be the perſpective or repreſentation of the original point H. It is in this way that objects are put in perſpective, namely, by finding the point where each optical ray meets the ſaid plane; and if the line or lines made in the ſuperficies of the optical cone or pyramid by this plane be thus found, they ſhall be the perſpective of the object, whether the object be a plane figure or a ſolid. Hence, the perſpective of a ſtraight line paſſing through the eye is a point, and the perſpective of a plane paſſing through the eye is a ſtraight line, viz. the common ſection of that plane with the one that is erected [3. 11.].

7. If the object be a plane figure, whether rectilineal or curvilineal, and if the plane erected between the eye and the object be parallel to the plane in which the

object is, the perſpective [16. & 10. 11. 4. 1. Appoll. con. or 23. 1. Simſ.] ſhall be ſimilar to the object: alſo, if the plane erected cut the optical pyramid or cone ſubcontrarily, that is, ſo that the part cut off from a ſcalene pyramid or cone towards the eye, be ſimilar to the whole pyramid or cone, the perſpective ſhall be ſimilar to the object: See 5. 1. Appoll. con. or 24. 1. Simſ. The reaſon of this laſt appears from what has been already ſaid: for, becauſe [art. 3.] the apparent magnitude of an object is reciprocally proportional to its diſtance from the eye, it is manifeſt, that the optical pyramid or cone may be ſo cut by a plane, that unequal lines ſhall ſubtend equal viſual angles: and this happens when the part cut off from a ſcalene pyramid or cone towards the eye is ſimilar to the whole pyramid or cone.

The perſpective of a plane figure is not ſimilar to that figure in any other caſe; and it is manifeſt, that the perſpective of a ſolid, taken upon any one plane, cannot be ſimilar to the ſolid whoſe perſpective it is.

Definitions. Fig. 2.

8. Def. 1. The plane NKLM, which is parallel to the horizon, and upon which the object to be put in perſpective is ſituated, is called the *geometrical plane.*

9. Def. 2. The plane ADFE, which is at right angles to the geometrical plane KM, between the eye O and the object, and in which the repreſentation of the object is found by means of the viſual rays, is called the *perſpective plane.*

10. Def. 3. AE, the common ſection of the geometrical and perſpective planes KM, AF, is called the *terreſtrial* or *baſe line.*

11. Def. 4. The plane DFQP, which paſſes through the eye O, and is parallel to the geometrical plane KM, is called the *horizontal plane.*

12. Def. 5. DF, the common ſection of the horizontal and perſpective planes DQ, AF, is called the *horizontal line.* The horizontal line [16. 11.] is parallel to the terreſtrial line.

13. Def. 6. The plane HVOG, which paſſes through the eye O, and is at right angles to each of the planes KM, AF, DQ,

viz. the geometrical, the perſpective, and the horizontal planes, is called the *vertical plane.*

14. Def. 7. OV, the common ſection of the horizontal and vertical planes DQ, HO, is called the *line of diſtance*; becauſe that part of it which is intercepted between the eye O, and the perſpective plane AF, is the perpendicular diſtance of the eye from the perſpective plane [19. 11.]; and if the perpendicular from the eye to the geometrical plane meet it in G, OV is [16. 11. & 34. 1.] equal to GH.

15. Def. 8. The point V, where the line of diſtance OV meets the perſpective plane, which point is in the horizontal line DF, is called the *principal point.*

16. Def. 9. VH, the common ſection of the vertical and perſpective planes HO, AF, is called the *line of the height*, becauſe it is equal to OG, the perpendicular height of the eye O above the geometrical plane KM.

PROP. I. THEOR. *Fig.* 3.

17. *If a ſtraight line* HC *be drawn from any point* H *in the geometrical plane* RM, *to*

to the point C, *where the perpendicular let fall from the eye* O *meets the geometrical plane, meeting the terreſtrial line* DE *in* L, *the perſpective of the point* H *ſhall be in the ſtraight line drawn from the point* L *at right angles to the geometrical plane.*

Let DB be the perſpective plane, and let the viſual ray HO meet this plane in G. Becauſe HO is the viſual ray of the point H, and that DB is the perſpective plane, the perſpective of the point H is in G, where HO meets DB [art. 6.]: but the three ſtraight lines HO, HC, and CO are [2. 11.] in one plane; and this plane is at right angles to the geometrical plane RM, becauſe OC is at right angles to RM [18. 11.]; and the perſpective plane DB [def. 2.] is alſo at right angles to the geometrical plane RM; wherefore their common ſection LG is [19. 11.] at right angles to RM. In like manner, if any other point K in the geometrical plane and C be joined by a ſtraight line, meeting the terreſtrial line in P, it is ſhown, that the perſpective of the point K is in the ſtraight line drawn from P at right angles to the geometrical plane RM. Q. E. D.

PROP. II. THEOR. *Fig.* 3.

18. *Every thing remaining as in the preceding propoſition,* HL *and* LC, *the ſum of the perpendicular diſtances of any point* H *in the geometrical plane, and* [def. 7.] *of the eye from the perſpective plane, is to* HL, *the perpendicular diſtance of* H *from the perſpective plane, as* CO, *the perpendicular height of the eye above the geometrical plane, to* LG, *the perpendicular height of the perſpective of the point* H.

Caſe 1. Let HL and LC, that is, HC, be at right angles to the perſpective plane DB; HC is therefore [def. 3. 11.] at right angles to the terreſtrial line DE. And becauſe OC and GL are [1. of this] each of them at right angles to the geometrical plane RM, they are [6. 11.] parallel; therefore [4. 6] CH is to HL, as CO to LG.

Caſe 2. But if K be the point in the geometrical plane, (the eye remaining as before), KC is not at right angles to the terreſtrial line DE. Let KC meet DE in P: if PF be drawn in the perſpective plane at right angles to the geometrical plane RM, it ſhall [1. of this] meet the viſual ray KO; let it meet in F, and draw KQ at right angles to the terreſtrial line DE, and let CL be as in the former caſe. Becauſe CL and KQ are each of them at right angles to DE, they are [28. 1.] parallel, and [29. 1.] the alternate angles PCL, PKQ, as alſo [15. 1.] the vertical angles CPL, QPK, are equal; wherefore the triangles CPL, QPK, are equiangular, and [4. 6.] CP is to PK, as CL to KQ: by compoſition, CK is to KP, as CL and KQ together to KQ: But as CK to KP, ſo [4. 6.] is CO to PF; wherefore CL and KQ together is to KQ, as CO to PF. Q. E. D.

Cor. 1. Hence, if V be the principal point, and VL the line of the height in the perſpective plane, and if VB be taken on the horizontal line AB, equal to the line of diſtance OV or CL, towards the ſame parts of VL with Q, and if QS be taken on the terreſtrial line towards L, equal to KQ, and if BS be joined, BS ſhall meet VQ in F, the perſpective of the point K. Becauſe CL and KQ together is to KQ, as CO or LV to PF; and that VB is equal to CL, and QS to KQ; therefore VB and SQ together is to SQ, as LV to PF.

Cor. 2. Hence, likewiſe the perſpective of any ſtraight line KQ, which is parallel to the line of diſtance OV, ſhall, if produced, paſs through the principal point V. Alſo the perſpective of any line, if produced, ſhall paſs through the point where a line drawn from the eye, parallel to the former, meets the perſpective plane; and therefore the perſpectives of any number of parallel ſtraight lines, which, when produced, meet the perſpective plane, ſhall meet in a point.

PROP. III. PROB. *Fig.* 4.

19. *To find the perſpective of any given point* E, *ſituated in the geometrical plane, when the perſpective plane* ACDB *coincides with the geometrical plane.*

Let CD be the terreſtrial line [def. 3.], and AB the horizontal line [def. 5.]; let V be the principal point; VO, being drawn at right angles to AB and CD, is the line of the height [def. 8. & 9.]. From the principal point V, cut off VA from the horizontal line, towards the ſame parts of VO that E is, equal to the eye's diſtance from the perſpective plane [def. 7.]; from E, draw EK at right angles to the terreſtrial line CD, meeting it in K; join VK; make KH upon the terreſtrial line towards O equal to KE, and join AH, meeting VK in G; G is the perſpective of the point E.

From G, draw GF at right angles to CD, meeting it in F. Becauſe VA and KH are [def. 5. of this] parallel, the alternate angles AVG, GKH, and VAG, GHK, are [29. 1.] equal, and the vertical angles AGV, KGH, are [15. 1.] equal; wherefore [4. 6.] VG is to GK, as VA to KH: by compoſition, VK is to KG, as VA and KH together to KH: but becauſe VO and GF are at right angles to CD, VK is to KG, as VO to GF; wherefore VA, and KH or KE, is to KE, as VO to GF; that is, the ſum of the perpendicular diſtances of the eye, and of the point in the geometrical plane from the perſpective plane, is to the perpendicular diſtance of the ſame point from the perſpective plane, as the height of the eye to GF; wherefore G is [cor. 1. 2. of this] the perſpective of the point E. Which was to be found.

Equal parts, anſwering to the diſtance of the object, or of the eye, &c. are to be taken from a ſcale of equal parts, as deſcribed art. 3. 4. 6. ſect. 3. 1.

The perſpective of any other point L is found in the ſame manner. Thus, if VB be taken on the ſame ſide of V that L is of O, equal to the diſtance of the eye from the perſpective plane, and if MN be taken towards O, equal to ML the perpendicular diſtance of the point L from the terreſtrial line CD; then joining VM, BN meeting in P, P ſhall be the perſpective of the point L: for if a perpendicular be let fall from P to the terreſtrial line CD, the demonſtration ſhall be the ſame with that of the preceding.

21. Example 1. fig. 5. To put in perſpective a triangle EFG.

Let fall perpendiculars from the angular points E, F, G, meeting the terreſtrial line CD in Q, H, P; let V be the principal point, and join VQ, VH, VP: and if part of the triangle EFG be on each ſide of the vertical plane, that is, on each ſide of the perpendicular drawn from V to CD, take VA, VB, each equal to the line of diſtance, and QK, HL, PK, each equal to the perpendiculars EQ, FH, GP: if AK, BL, BK, be joined, meeting VQ, VH, VP, in N, O, M, theſe points ſhall be the perſpectives of E, F, G, by what was already demonſtrated; and if NO, OM, and NM, be joined, the triangle NOM ſhall be the perſpective of the triangle EFG.

22. Example 2. fig. 6. To put in perſpective a ſquare EFGH, which is on one ſide of the vertical plane.

Draw the perpendiculars EK, FL, HM, GN, as before; make each of the ſtraight lines KC, LO, MP, NL, equal to each of theſe perpendiculars; join the principal point V, and each of the points where the perpendiculars meet CD; and becauſe the ſquare EFGH is on one ſide of the vertical plane, take VB on the ſame ſide equal to the line of diſtance, and join BC, BO, BP, BL, meeting VK, VL, VM, VN, in the points Q, R, T, S, and join theſe points, the figure QRST is the perſpective of the ſquare EFGH. In like manner, any rectilineal figure whatever is put in perſpective.

23. Example 3. fig. 7. To put in perſpective a ſquare EFGH, which is equally divided by the vertical plane, through

whoſe diagonal FH the vertical plane paſſes, and one of whoſe angular points F is in the terreſtrial line CD.

Produce HG, HE, meeting the terreſtrial line CD in K, M; take VA, VB, each equal to the line of diſtance on each ſide of the principal point V; join AF, AK, BF, BM, meeting AF, AK, in O, Q, P; the figure OFQP is the perſpective of the ſquare EFGH. For if the perpendiculars GL, EN, be drawn, and VL, VN, joined, as in the former examples, FHK is half a right angle; therefore FKH is half a right angle, and [6. 1.] FK is equal to FH. And becauſe each of the angles LFG, LKG, is half a right angle, and that the angles at L are right angles, FL [26. 1.] is equal to LK.

24. Example 4. fig. 8. To put in perſpective a ſquare EFHG, which is equally divided by the vertical plane, one of whoſe ſides EF coincides with the terreſtrial line CD.

Let V be the principal point; join VE, VF, and take the points of diſtance A, B, as above; join AF, BE, meeting VE, VF, in K, L, and join KL; the figure EKLF is the perſpective of the ſquare EFHG.

It is to be obſerved in ſuch caſes as the two laſt examples, when any number of points of the object are in a ſtraight line which is parallel to the terreſtrial line, as E, G, fig. 7. or G, H, fig. 8. that when O, fig. 7. the perſpective of E, or K, fig. 8. the perſpective of G, is found, the perſpective of G, fig. 7. or of H, fig. 8. may be found, by drawing through O and K ſtraight lines parallel to the terreſtrial line CD, meeting VL, VK.

25. Example 5. fig. 9. To put in perſpective a circle EFGH, which is equally divided by the vertical plane.

Take any number of points K, L, M, N, in the circumference, and from each of theſe points draw a perpendicular to the terreſtrial line CD; join the principal point V, and each of the points where theſe perpendiculars meet the terreſtrial line: let B be one of the points of diſtance, and make ST equal to SF, and join BT, meeting VS, *e*, the interſection of BT, VS, is the perſpective of F. In like manner, if SC be made equal to SH, *f*, the interſection of BC, VS, ſhall be the perſpective of H; the perſpective of each point, taken in the ſemicircle FGH, is found the ſame way.

If a parallel to the terreſtrial line be drawn through the perſpective of each of the points K, L, G, M, N, the lines drawn from V on the other ſide of the vertical plane ſhall be cut in the correſponding perſpective points of the other ſemicircle FEH; and if an indefinite number of theſe points be found, and joined, the curve thence ariſing ſhall be the perſpective of the circle.

In other caſes, the perſpective of each point in the whole circumference muſt be found ſeparately.

The perſpective of the ſemicircle EHG, which is more remote from the eye, has leſs curvature than that of the other ſemicircle EFG, which is nearer the eye. For becauſe the ſtraight line K*d*, which is equal to N*b*, being nearer the eye [21. 1.], ſubtends a greater viſual angle than N*b* does, and that the perſpectives of theſe ſtraight lines are to one another as the viſual angles; the perſpective of the arc NH*b* being diminiſhed more than the perſpective of the arc KF*d*, muſt approach nearer to a ſtraight line. Hence the perſpective of a circle, which is in the geometrical plane, cannot be the ſame ellipſis throughout, if the perſpective plane be at right angles to the geometrical plane. It is, however, two ſemi-ellipſes, having a common greater axis, whoſe leſſer axes are to one another as *ec* in the fig. to *cf*; which is eaſily ſhewn from the nature of the ſection of a ſcalene cone, by a plane at right angles to the baſe.

Of the Perſpectives of Solids.

PROP. IV. THEOR. *Plate 5. fig.* 1.

26. *Let* B *be a point above the geometrical plane* NP, *and let the perpendicular drawn from* B *to the geometrical plane meet it in* A; *the ſum of the perpendicular diſtances of the point* A, *and of the eye from the perſpective plane* DM, *is to the diſtance of the eye from the perſpective plane, as the perpendicular* AB *to its perſpective.*

Let the perpendicular drawn from the eye O to the geometrical plane meet it in T, and join OA, OB, meeting the perſpective plane DM in G, H; GH being joined, is the perſpective of AB [art. 2. & 4. of this]: and if TA be joined, meeting the terreſtrial line DE in S, the ſtraight line drawn from S at right angles to the geometrical plane NP ſhall paſs through the points G and H [1. of this].

Caſe 1. Let AT be at right angles to the terreſtrial line DE. Becauſe TO and SH are [6. 11.] parallel, AT is to TS [2. 6.], as AO to OG; that is [4. 6.], as AB to GH; wherefore AT is to TS, as AB to GH.

Caſe 2. Next let F be the point above the geometrical plane, and C the point where the perpendicular drawn from F to the geometrical plane meets it; the eye O remaining as in the firſt caſe, CT cannot be at right angles to DE: join OC, OF, meeting the perſpective plane in K, L; let CT meet DE in R, and draw CQ at right angles to DE, meeting it in Q. Becauſe TS and CQ are each of them at right angles to DE, they are [28. 1.] parallel, and [29. 1.] the alternate angles CQR, RST, and RCQ, RTS, are equal; as alſo [15. 1.] the vertical angles CRQ, SRT; wherefore [4. 6.] CR is to RT, as CQ to TS: by compoſition, CT is to TR, as CQ and TS together to TS: but CT is to TR [2. 6.], as CO to OK, that is [4. 6.], as CF to KL; wherefore CQ and TS together is to TS, as CF

to

L. But CQ and TS are the perpen- ar diſtances of the object and of the from the perſpective plane, and KL is perſpective of CF. Q. E. D.

PROP. V. PROB. *Fig.* 2.

To find the perſpective of a given ſtraight line, which is at right angles to the geometrical plane, meeting it in a given point, when the perſpective plane coincides with the geometrical plane.

Let AB be the horizontal, and CD the rreſtrial line; let M [3. of this] be the rſpective of the point where the perpen- cular meets the geometrical plane, and let e ſtraight line O be equal to the perpen- cular whoſe perſpective is ſought,

From any point E, in the terreſtrial line CD, draw EF at right angles to CD, meeting the horizontal line AB in F: make FL equal to the perpendicular diſtance of the eye from the perſpective plane, and LV equal to the perpendicular diſtance of the line which is the object from the perſpective plane; make EG equal to O, and join VG, VE; from M draw MH parallel to CD, meeting VE in H, and join LH, (which [1. of this] is parallel to EF), meeting VG in K; from K draw KN parallel to HM, and from M draw MN parallel to HK, meeting KN in N; MN is the perſpective required. Becauſe HL is parallel to EF, VF [2. 6.] is to FL, as VE to EH, or as VG to GK, that is [4. 6.], as EG or *O* to HK; therefore VF is to FL, as O to HK: wherefore [4. of this] HK or [34. 1.] MN is the perſpective required, and N is the perſpective of the point above the geometrical plane. Which was to be found.

Cor. The perſpective height MN *may* be found by means of any other point P in the horizontal line, as well as V. Join PE, PG, and let MH meet PE in Q; from Q draw QR parallel to EF, meeting PG in R. Becauſe VF is to FL, as VE to EH, that is [2. 6.], (becauſe QH, PV, are parallel), as PE to EQ, or as PG to GR, that is [4. 6.], as EG to QR; wherefore EG has the ſame ratio to QR that it has to HK, and [9. 5.] QR is equal to HK. Q. E. D.

PROP.

PROP. VI. PROB. *Fig.* 3. & 4.

28. *To find the perſpective of a ſtraight line which is inclined to the geometrical plane, meeting it in a given point, the perpendicular let fall from its extremity above the geometrical plane, and the point where that perpendicular meets the geometrical plane being given.*

Find [3. of this] L, the perſpective of the point where the inclined line meets the geometrical plane; and M, the perſpective of the point where the perpendicular meets the geometrical plane: take any point E in the terreſtrial line CD, and draw EG at right angles to CD, making it equal to the perpendicular: take any point V in the horizontal line AB, and join VE, VG: draw MH parallel to CD, HK parallel to EG, KN parallel to MH, and MN parallel to HK, meeting KN in the point N: N is the perſpective of the extremity which is above the geometrical plane [5. of this]; and LN being joined, is the perſpective required.

If the inclined line whoſe perſpective is ſought be in the vertical plane, fig. 3. the three points L, M, N, ſhall be in a ſtraight line; but if it be not in the vertical plane, fig. 4. the three points L, M, N, ſhall not be in a ſtraight line.

29. Example 1. fig, 5. To put in perſpective a given cube or parallelepiped GN, whoſe baſe is FGHK, and altitude the ſtraight line GL.

Let CD be the terreſtrial, and AB the horizontal line, V the principal point. Find [3. of this] QRST, the perſpective of FGHK; from any point D in CD, draw DP at right angles to CD, making it equal to GL, the altitude of the ſolid GN; take any point E in AB, and join ED, EP; and from each of the points Q, R, S, T, draw ſtraight lines parallel to CD, meeting ED; from the points where theſe parallels meet ED, draw ſtraight lines parallel to DP, meeting EP; and from the points where theſe laſt parallels meet EP, draw ſtraight lines parallel to CD; laſtly, from T, S, R, Q, draw ſtraight lines parallel to DP, meeting the laſt parallels to CD in U, Z, X, Y; the figure TY [5. of this] is the perſpective of the ſolid GN; the plane RT is the perſpective of the plane KG; the plane TX, of GO; the plane TZ, of GM; the plane QY, of FN; the plane SY, of HN; and the plane UY, of LN. Which was to be done.

30. Example 2. fig. 6. To put in perſpective a given pyramid KFGHO.

Let CD be the terreſtrial, and AB the horizontal line. Find LMQR, the perſpective of FKHG, which is the baſe of the pyramid: find alſo the point N where the perpendicular drawn from O, the vertex of the pyramid, meets the baſe; and find S, the perſpective of the point N; draw DP, as in the laſt example, equal to the perpendicular ON; by means of which ST, the perſpective of ON, is found as above; join TL, TM, TQ, TR, the figure LMQRT is the perſpective of the pyramid KFGHO [6. of this]. Which was to be done.

31. Example 3. fig. 7. To put in perſpective three given walls, one of which is parallel to the perſpective plane, and the other two parallel to the vertical plane, and on oppoſite ſides of it.

Let AB be the horizontal, and CD the terreſtrial line, V the principal point, A, B, the points of diſtance. Find [3. of this] KFHQ, the perſpective of the baſe on which the walls ſtand; make KC, QD, equal to the thickneſs of the walls, and

KX, C*a*, QY, D*d*, (at right angles to CD), equal to the height of the walls; join XV, YV; and from F, H, draw FE, HG, at right angles to FH, meeting XV, YV, in E, G, and join EG; EFHG is the perſpective of the wall which is parallel to the perſpective plane; XKFE, YQHG, are the perſpectives of the other two walls. Which was to be done.

32. Example 4. fig. 7. To find the perſpective of a door with its poſts, in the wall whoſe perſpective is XKFE.

Make KN upon the terreſtrial line equal to the diſtance of the neareſt poſt from K, NI equal to the breadth of the poſt, IL equal to the breadth of the door, LM equal to the breadth of the other poſt; and from theſe points, draw ſtraight lines to the neareſt point of diſtance A, meeting KV in *n*, *i*, *l*, *m*; and from *n*, *i*, *l*, *m*, draw ſtraight lines parallel to KX; make KP equal to the height of the poſt, and KO equal to the height of the door, and join PV, OV, meeting *nb*, *mb*, in *b*, *b*, and *if*, *le*, in *f*, *e*; *f i le*, is the perſpective of the door; *ib*, *lb*, are the perſpectives of the poſts. Which was to be found.

33. Example 5. fig. 7. To find the perſpective of a door with its poſts, in the wall whoſe perſpective is EFHG.

Make KM equal to the diſtance of the poſt from the angular point whoſe perſpective is the point F; make MR equal to the breadth of the poſt, RT equal to the breadth of the door, TS equal to the breadth of the other poſt, and join VM, VR, VT, VS, meeting FH in 1, 2, 3, 4; from 1, 2, 3, 4, draw ſtraight lines parallel to KX; make KP equal to the height of the poſt, and KO equal to the height of the door, and join PV, OV, meeting EF in Z, 5; from Z, 5, draw ſtraight lines parallel to CD, meeting 1*r*, 2 6, in *r*, 6, and 4*t*, 3 7 in *t*, 7; 6 2, 3 7 is the perſpective of the door; 2*r*, 3*t*, are the perſpectives of the poſts. Which was to be found.

34. Example 6. fig. 7. To find the perſpective of a window in the wall, whoſe perſpective is GHQY.

Let Qβ be the depth of the window in the wall; make QU equal to the diſtance of the window from Q, and US equal to its breadth; from S, U, draw ſtraight lines to the neareſt point of diſtance B; make β8 equal to the height of the bottom of the window, and β9 equal to the height of its top; and from 8, 9, draw ſtraight lines to the principal point V: laſtly, from the points where SB, UB, meet HQ, draw ſtraight lines parallel to QY, meeting the ſtraight lines drawn from 8 and 9 to V; the figure *pq* is the perſpective of the window. Which was to be found.

In this manner any building may be put in perſpective.

Of Shadows.

35. It is manifeſt, fig. 8. that if AD be the perſpective of any ſtraight line at right angles to the geometrical plane, L, the perſpective of the luminous point, and M, the perſpective of the point where the perpendicular drawn from L meets the geometrical plane, the perſpective of the ſhadow of AD ſhall be terminated in the point G, where the ſtraight line drawn from L through A, meets the ſtraight line drawn from M through D. In like manner, the ſhadow of BE is terminated in H, where LB and ME meet.

36. Example 1. fig. 8. To find the perſpective of the ſhadow of a priſm, whoſe

perſpective is AFBD, ſtanding upon the triangular baſe FDE, and whoſe inſiſting ſtraight lines are at right angles to the baſe.

Let L be the perſpective of the luminous point, and M the perſpective of the point where the perpendicular drawn from the luminous point meets the geometrical plane. From L draw ſtraight lines through A, B, C, the angular points of the priſm that are above the geometrical plane; and from M, draw ſtraight lines through F, D, E, the angular points of the priſm that are in the geometrical plane. If the points where theſe laſt ſtraight lines meet the former be joined, the ſhadow of the priſm ſhall be determined. Becauſe [art. 35.] DG is the ſhadow of AD, EH the ſhadow of BE, and FK the ſhadow of CF, the figure EFKGH is the perſpective of the ſhadow of the priſm AFBD: but FDE, the baſe of the priſm, is not conſidered as a part of the ſhadow; therefore the ſhadow required is EDGH.

If the inſiſting ſtraight lines AD, BE, CF, be not at right angles to the geometrical plane or baſe of the figure, the ſtraight lines LA, LB, &c. are to be drawn as before; then the perſpective of the points where the perpendiculars drawn from A, B,

&c. meet the geometrical plane, muſt be found, and ſtraight lines muſt be drawn through theſe laſt points from M, meeting LA, LB, &c. Laſtly, ſtraight lines are drawn from M to the angular points of the baſe of the priſm. If the points ſo found be joined, the ſhadow ſhall be determined. In this manner, the perſpective of the ſhadow of any ſolid is found.

37. It is manifeſt, that the angles of which DAG, EBH, are the perſpectives, contained by the perpendiculars AD, BE, and the rays LG, LH, are the compliments of the elevation of the luminous point L at the points G, H, or A, B, [def. 5. part 1. ſect. 1.]. If the rays proceeding from L diverge conſiderably, as the rays of a candle, &c. the ſhadow becomes broader in proportion to the diſtance from the object, or from the luminous point. Likewiſe, if the point A be nearer L than B is, the angle DAG ſhall be leſs than the angle EBH, as is manifeſt.

38 Example 2. plate 6. fig. 1. To find the perſpective of the ſhadow of a parallelepiped, illuminated by the ſun, whoſe perſpective is the figure AEDG.

Becauſe the rays of the ſun proceed from ſo great a diſtance, they may be conſidered as parallel: therefore, through the angular points of the baſe of the figure H, E, F, G, draw ſtraight lines parallel to the terreſtrial line; and at the point A, in the perpendicular AG, make the angle GAK equal to the complement of the ſun's elevation above the geometrical plane, and let AK meet GK in K; through D and B, draw ſtraight lines parallel to AK, meeting the other parallels to the terreſtrial line in M and L; the figure HLKMF determines the ſhadow.

If the elevation of the ſun be forty-five degrees, it is manifeſt, that the angle GAK is found, by making GK equal to GA.

In theſe examples, the luminous point or body is ſuppoſed to be in the perſpective plane: but by the following general propoſition, the ſhadow is found in whatever ſituation it is.

PROP. VII. *Fig.* 2. & 3.

39. *The diſtance of the ſun from the vertical plane, (meaſured upon the horizon), and his elevation above the horizon being each of them given, to find the poſition and magni-*

tude of the ſhadow of an object, which is at right angles to the geometrical plane, and whoſe perſpective is the ſtraight line LK.

Caſe 1. When the ſun is beyond the perſpective plane, let AB, fig. 2. be the horizontal, and NG the terreſtrial line; V the principal point, and VE the line of diſtance; draw VG at right angles to AB, and make VC equal to VE: at the point C, make the angle VCD, towards the ſun, equal to the diſtance of the ſun from the vertical plane; or, which comes to the ſame thing, make the arc VH, deſcribed from the centre C, with the diſtance CV, equal to the arc of the horizon intercepted between the perſpective vertical plane and the vertical circle in which the ſun is, and join CH, meeting AB in D: D is the perſpective of the point where the perpendicular drawn from the ſun to the geometrical plane meets this plane.

Make DB equal to DC; at the point B, make the angle DBF equal to the ſun's elevation above the horizon, and let BF meet the ſtraight line drawn from D at right angles to AB in F: F is the perſpective of the point where the ſun is; and if ſtraight lines be drawn from D to K, and from F to L, meeting one another in M, KM ſhall be the perſpective of the ſhadow of LK.

For (while the perſpective plane ABGN is at right angles to the geometrical plane) if the triangle DCV be turned about upon the ſtraight line DV, until it become parallel to the geometrical plane, that is, until the point C be elevated above the geometrical as much as the eye is; then, becauſe the plane in which are the ſtraight lines DC, DF, is [18. 11.] at right angles to the plane of the triangle DCV, and to the geometrical plane, CD is [16. 11.] parallel to the common ſection of the plane paſſing through DC, DF with the geometrical plane: and becauſe the angle VCD is equal to the diſtance of the ſun from the vertical plane, the ſhadows of all ſtraight lines that are at right angles to the geometrical plane are in the geometrical plane [9. 11.] parallel to CD; therefore the ſhadow of LK, in the geometrical plane, is parallel to CD; and therefore [cor. 2. 2. of this] the perſpective of the ſhadow of LK, if produced, ſhall paſs through the point D.

Other things remaining as before; let the triangle FBD be turned about upon FD, until DB coincide with DC, then the point B ſhall coincide with C, and the ſtraight line FB with FC; wherefore the angle DCF ſhall be equal to the angle DBF, that is, to the ſun's elevation: and becauſe CD, as has been ſhown, is parallel to the ſtraight line in the geometrical plane, whoſe perſpective is KM, and that FD [6. 11.] is parallel to the ſtraight line whoſe perſpective is LK, the plane paſſing through the triangle FCD is [15. 11.] parallel to the plane paſſing through the triangle whoſe perſpective is the triangle LMK; and the angle whoſe perſpective is LMK [art. 37.] is equal to the angle DCF; wherefore [conv. 10. 11.] the ſtraight line whoſe perſpective is ML, is parallel to FC; therefore ML produced ſhall [cor. 2. 2. of this] paſs through the point F.

Caſe 2. fig. 3. When the ſun is on the ſame ſide of the perſpective plane with the eye, let V be the principal point; make VC equal to the line of diſtance VE, and the angle VCD equal to the diſtance of the ſun from the vertical plane: but in this caſe, the angle VCD muſt be made on the ſide of VG, which is oppoſite to the ſun; that is, towards the right hand, if the ſun is towards the left; let CD meet AB in D, and join DK: DK is the poſition of the ſhadow of LK.

Make DB equal to DC; and, in this caſe, make the angle DBF, on the ſame ſide of AB that the point C is, equal to the ſun's elevation; let BF meet the ſtraight line drawn from D, at right angles to AB in F, and join FL, meeting DK in M: KM is the perſpective of the ſhadow of LK. The demonſtration is the ſame with that of the firſt caſe. Which was to be done.

In this manner, the perſpectives of the ſhadows of all other ſtraight lines at right angles to the geometrical plane are found; and by means of theſe, the perſpective of the ſhadow of any ſolid is [art. 36. & 38.] found.

S E C T. II.

Projection of the Sphere.

1. PRojection of the ſphere is a particular caſe of perſpective, by which the repreſentations or perſpectives of the ſeveral circles of a ſphere are deſcribed upon the plane of any one of its great circles.

2. Def. 1. The great circle of the ſphere upon whoſe plane the other circles are projected, is called *the primitive circle*. It correſponds to the perſpective plane in perſpective [def. 2. per.]. The ſtraight line drawn from the eye at right angles to the plane of this circle paſſes through its centre. In the ſtereographic projection of the ſphere, the eye is ſuppoſed to be in the pole of the primitive circle. The primitive circle is deſcribed with the chord of ſixty degrees. See art. 10. 11. 12. ſect. 3. 1.; and the ſines, tangents, and ſecants, afterwards mentioned, are taken from a ſcale of the ſame radius.

3. Def. 2. All circles of the ſphere whoſe planes are parallel to the plane of the primitive circle, are called *direct circles*.

Hence, all direct circles are projected into circles that are equal to them [art. 7. per.].

4. Def. 3. All circles whoſe planes are at right angles to the plane of the primitive circle, are called *right circles*.

Hence, all right circles are projected into ſtraight lines [art. 6. per.] equal to their diameters.

5. Def. 4. All other circles of the ſphere are called *oblique circles.*

6. Def. 5. A *ſpherical angle* is the inclination of two great circles meeting one another in the ſuperficies of the ſphere, which is the ſame thing with the inclination of the planes in which theſe great circles are; that is, the angle contained by the ſtraight lines touching the circles at the point where they meet [11. 1. Theod. 18. 3. & def. 6. 11.]. A ſpherical angle is meaſured by the arc of a great circle paſſing through the poles of the great circles that contain the ſpherical angle, between the points where it meets them; and theſe points are a quadrant diſtant from the ſpherical angle [13. 1. Theod.].

7. Def. 6. The common ſection of a circle, paſſing through the eye and through the poles of the primitive, and of the circle to be projected with the primitive circle, is called *the line of meaſures.* This circle, which [conv. 13. 1. Theod.] is at right angles to the primitive circle, and to the circle to be projected, correſponds in perſpective to the vertical plane [def. 6. per.].

PROP. I. THEOR. *Plate* 6. *fig*. 4.

8. *The parts of the diameter* AC, *into which* [art. 4.] *a right circle* ABC *is projected, are each of them equal to the half tangents of the arcs* BF, BG *which they repreſent.*

For if the eye be in D, the pole of the great circle AC, its other pole B ſhall appear [10. 1. Theod.] in E the centre; join DF, DG, meeting AC in H, K: H, K, are the perſpectives of the points F, G; and therefore EH, EK, are the perſpectives of the arcs BF, BG: join FE, GE; and becauſe the angle FEB [20. 3.] is double of the angle HDE, the arc BF, which meaſures the angle FEB, is double of the arc which meaſures the angle HDE, becauſe DE is equal to EB: but EH is [def. 3. ſect. 1.] the tangent of the arc, which is the meaſure of the angle HDE; therefore EH is the half tangent of the arc BF. In like manner, it is ſhewn, that EK is the half tangent of the arc BG. Q. E. D.

Cor. 1. Hence, if the ſemicircle ABC be divided into its degrees, and if ſtraight lines be drawn from D to each point of diviſion,

viſion, theſe ſtraight lines ſhall divide AC into its degrees.

Cor. 2. And if parallel leſſer circles KLM, HNO, are to be projected upon the plane of the circle ABCD, the diſtances EK, EH, with which they muſt be deſcribed, are found by making the arcs AG, AF, equal to the diſtances of the circles KLM, HNO, from the circle ABCD, and joining DG, DF, meeting AC in K, H; or, which is the ſame thing, making EH, EK, equal to the half tangents of BF, BG.

Cor. 3. Alſo, fig. 5. the diameters of any leſſer circles KLO, HNM, which are within the circumference of ABC, are found by making the arcs AG, AP, equal to the diſtances of the leſſer circles from ABCD towards A, and CQ, CF, equal to their diſtances from it towards C, and joining DG, DP, DQ, DF, meeting AC in K, H, M, O; KO, HM, are the diameters of the leſſer circles KLO, HNM. And if E, the centre of ABCD, is within the circumference of a leſſer circle, as KLO, it is manifeſt, that KO, the diameter of the leſſer circle KLO, is equal to the ſum of the half tangents of the arcs BG, BF, on each ſide of B, the other pole of AC. But if E, the centre of ABCD, is without the circumference of a leſſer circle HNM, HM is equal to the difference of the half tangents of the arcs BQ, BP.

PROP. II. THEOR. *Fig.* 6.

9. *When the eye is in the pole of the primitive circle, all oblique circles of the ſphere are projected into circles.*

Though the truth of this propoſition appears from what was ſaid [art. 7. per.], it is perhaps proper to give the common demonſtration of it here, which is but a particular caſe of the propoſition of Appollonius there referred to; for the rays proceeding from each point of the circumference of the oblique circle to the eye in the pole of the primitive, are in the ſuperficies of a cone, whoſe baſe is the oblique circle; and this cone is manifeſtly cut ſubcontrarily by the primitive circle.

Let EF be any oblique circle, and let the eye be in D, the pole of the primitive circle AC; and let B be its other pole. If DE, DF, be joined, meeting AC in G, L, the ſtraight line GL, (which [cor. 3. 1. of this] is equal to the ſum of the half tangents of the arcs BE, BF), is the projection of the ſtraight line EF, which is the diameter of the oblique circle EF. Let the cone, whoſe baſe is the oblique circle EF, and vertex the point D, be cut by a plane parallel to the baſe, the ſection NHMQ made in the ſuperficies of the cone by this plane ſhall be a circle [art. 7. per.]: let this circle meet the plane of the primitive in H, and draw HP at right angles to GL, meeting it in P; through P, draw NM, a diameter of the circle NHMQ, parallel to EF, the diameter of the baſe of the cone: let O be the centre of the circle ABCD, and join EB. Becauſe the primitive circle AC is [conv. 13. 1. Theod.] at right angles to ABCD, HP [def. 3. & 4. 11.] is at right angles to NM; therefore [35. 3.] the ſquare of HP is equal to the rectangle NPM: but becauſe the angles DOG, DEB, [10. 1. Theod. & 31. 3.] are right angles, and that the angle EDB is common to the two triangles DGO, DEB, the angle DGO [32. 1.] is equal to the angle EBD; that is [21. 3.], to the angle EFD, or [29. 1.] to the angle PML: wherefore the angle NGP is equal to the angle PML, and [15. 1.] the vertical angles GPN, MPL, are equal, and the triangle GNP is [32. 1.] equiangular to the triangle PLM; therefore [4. 6.] NP is to GP, as to PL to PM; and [16. 6.] the rectangle GPL is equal to the rectangle NPM, that is, to the ſquare of HP; therefore [conv. 35. 3.] the point H is in the circumference of a circle whoſe diameter is GL. In like manner, any other point in the ſection made by the primitive AC in the ſuperficies of the cone, may be ſhewn to be in the circumference of the circle in which H is. Q. E. D.

Cor. 1. Hence, fig. 5. the leſſer oblique circles KLO, HNM, are projected upon the plane of the primitive ABCD, by biſecting the diameters KO, HM, found as in cor. 3, 1. of this, in R, S.

Cor. 2. Hence a great circle, fig. 7. inclined to the primitive ABCD, and paſſing through the extremities of the diameter BD, is projected as follows. Draw the diameter AC at right angles to BD, (which is therefore the line of meaſures, def. 6. of this), and make the arc AF equal to the inclination of the circle to be projected to the primitive; join DF, meeting AC in G; and through the three points B, G, D, deſcribe [5. 4.] a circle; this circle ſhall be the projection required,

quired. The centre of the circle to be projected, however, is found more eaſily by means of the following propoſition.

PROP. III. THEOR. *Fig.* 7.

10. *The centre of the projection of a great circle is in the line of meaſures, and its diſtance from the centre of the primitive is equal to the tangent of that great circle's inclination to the primitive.*

Let AQ [art. 7.] be the line of meaſures, and make AF equal to the inclination of the great circle whoſe projection is GBQD to the primitive ABCD; join DF, meeting AQ in G; and from F, draw FK, a diameter of the primitive; join DK, meeting AQ in Q. If the eye be in D, GQ is the projection of the diameter FK; and if GQ be biſected in H, H ſhall be the centre of the projection GBQD. Join DH, meeting FK in L. Becauſe HD is equal to HQ, the angle EQD [5. 1.] is equal to the angle LDK; and becauſe EK is equal to ED, the angle EDQ is equal to the angle LKD; wherefore [32. 1.] the angle DLK is equal to the angle DEQ, and is therefore a right angle; therefore EL is at right angles to DH, and [8. 6.] the triangle DHE is ſimilar to the triangle EHL; therefore the angle EDH is equal to the angle LEH: but LEH [15. 1.] is equal to the angle FEA, which is the inclination of the circle whoſe projection is GBQD to the primitive; therefore the angle EDH is equal to that inclination, and EH [def. 3. ſect. 1. 1.] is the tangent of the angle EDH, DE being radius. Q. E. D.

Cor. 1. DH [def. 4. ſect. 1. 1.] is the ſecant of the angle EDH, or of the inclination.

Cor. 2. Hence a ſpherical angle of any given number of degrees may be deſcribed.

(1) If the ſpherical is to be deſcribed at the point B in the circumference of the primitive circle ABD, let E be the centre of the primitive; draw the diameter BD; draw another diameter AC at right angles to BD; make EH [art. 14. ſect. 3. 1.] equal to the tangent of the ſpherical angle to be deſcribed: from the centre H, with the diſtance HB, deſcribe the arc BG; the ſpherical angle ABG is the one required. The ſame thing is done by making BH equal to the ſecant of the inclination [art. 15. ſect. 3. 1.].

(2)

(2) If the ſpherical angle is to be deſcribed at any given point P, fig. 8. within the circumference of the primitive circle ABCD, let E be the centre of the primitive ABCD; through P, draw BD, a diameter of the primitive, and another diameter AC at right angles to BD; through the three points A, P, C, deſcribe a circle HPF, let G be the centre of this laſt circle, and draw a diameter of it HF at right angles to BD; make GK equal to the tangent of the ſpherical angle to be deſcribed at P; from the centre K, with the diſtance KP, deſcribe an arc LPM; the ſpherical angle LPC or APM is the one required. This is manifeſt from the preceding caſe: For after the circle HPF is deſcribed, (by making EG equal to the tangent of its inclination to ABCD), it is conſidered as a primitive circle; and the ſpherical angle APM is deſcribed at the point P in the circumference of it as in the preceding caſe. If the ſpherical angle at P is to be contained by PM, an arc of an oblique circle, and PE an arc of a right circle; the ſame conſtruction being made; make GK equal to the tangent of the complement of the ſpherical angle to be deſcribed; and deſcribe the arc LPM as before,

the ſpherical angle MPE is the one required; for the ſpherical angle MPE is the complement of the ſpherical angle APM. It is manifeſt, that a ſpherical angle at the pole or centre of the primitive circle is a rectilineal angle. Thus, fig. 4. the angle AEG is contained by AE, GE, arcs of right circles, and is meaſured by the arc AG of the primitive circle.

Cor. 4. The number of degrees of a ſpherical angle may be found by the converſe of theſe methods.

(1) If the ſpherical angle ABG, fig. 7. be at any point B in the circumference of the primitive circle, find E and H [25. 3.], the centres of the arcs AB, GB, that contain the ſpherical angle ABG, EH is equal to the tangent of the ſpherical angle ABG. In like manner, GK, fig. 8. is equal to the tangent of the ſpherical angle APM or LPC, and to the cotangent of the ſpherical angle MPE, contained by an oblique and a right circle.

The meaſure of a ſpherical angle ABG, fig. 7. may alſo be found upon the circumference of the primitive circle ABCD. Let E be the centre of the primitive, and draw the diameter BD; draw another diameter AC at right angles to BD, meeting the arc BG in G; join DG, meeting the primitive circle in F; the arc AF [cor. 2. 2. of this] is the meaſure of the ſpherical angle ABG.

PROP. IV. THEOR. *Fig. 7.*

11. *When the eye is in the pole of the primitive circle, all leſſer circles whoſe planes are at right angles to the plane of the primitive, or* [13. 1. Theod.] *through whoſe poles the primitive paſſes, are projected into circles.*

Let NK be a diameter of a leſſer circle, through whoſe poles A, C, the primitive ABCD paſſes: let the eye be in D, a quadrant diſtant from the poles of the leſſer circle, and join AC; join DN, DK, meeting AC in M, Q; MQ is the projection of the diameter NK, and MQ, NK, are placed ſubcontrarily. From D, draw DB, a diameter of the circle ABCD, it ſhall be at right angles to AC; let E be the centre of ABCD, and join KB. Becauſe DEQ is a right angle, and that DKB [31. 3.] is a right angle, and the angle BDK is common to the two triangles BDK, EDQ; the angle DQE is

equal

equal to the angle DBK, that is [21. 3.], to the angle DNK, and the angle NDQ is common to the two triangles NDK, MDQ; therefore the angle DKN is equal to the angle DMQ, and MQ is placed ſubcontrarily to NK; therefore the circle whoſe diameter is MQ, and whoſe plane is at right angles to the plane of ABCD, cuts the cone whoſe baſe is NK, and whoſe vertex is D, ſubcontrarily, and, as was ſhewn, [2. of this] makes the circumference of a circle in the ſuperficies of that cone. Q. E. D.

Cor. 1. Hence the projection of any leſſer circle, through whoſe poles C, A, the primitive ABCD paſſes, is deſcribed as follows. Make CN, CK, each of them equal to the diſtance of the leſſer circle from its neareſt pole C; from D, the extremity of the diameter of the primitive, which is at right angles to AC, draw DN, meeting AC in M; and through the three points N, M, K, [5. 4.] deſcribe a circle; it ſhall be the projection required.

Cor. 2. Hence, and prop. 2. if a cone whoſe vertex is any point in the ſuperficies of the ſphere, and whoſe baſe is a circle in the plane of the primitive, be cut by a plane parallel to the primitive, the ſection of the cone by this laſt plane ſhall alſo be a circle.

The centre of a leſſer circle to be deſcribed, however, is found more eaſily by means of the following propoſition.

PROP. V. THEOR. *Fig.* 7.

12. *The centre of the projection of a leſſer circle, through whoſe poles the primitive paſſes, is in the line of meaſures; and its diſtance from the centre of the primitive, is equal to the ſecant of the diſtance of the circumference of the leſſer circle from its neareſt pole.*

The ſame conſtruction remaining as in the preceding prop. and cor. 1. Let P be the centre of MNQK, which is the projection of the leſſer circle, which meets the primitive ABCD in N, K; join EN, NP, NQ, and NK, meeting AQ in O. Becauſe the angle DEK [20. 3.] is double of the angle DNK, and that the angle EKN [19. 11. & 29. 1.] or ENK [5. 1.] is equal to DEK, the angle ENO is double of the angle MNO, that is (becauſe MNQ [33. 3.] is a right angle, and that NO is at right angles to MQ) [8. 6.], double of the angle MQN: But the angle MPN [20. 3.] is alſo double

of

of MQN; wherefore the angle ENO is equal to the angle EPN; and [conv. 8. 6.] the triangle EPN is ſimilar to the right-angled triangle ENO: therefore the angle ENP is a right angle, and EP [def. 4. ſect. 1. 1.] is the ſecant of the angle NEP, or of the arc NC. Q. E. D.

The centre P may alſo be found by making NP equal to the tangent of the arc NC.

Cor. From this, and from prop. 3. it is manifeſt, that the angle contained by two ſtraight lines drawn from the point where the projection of any circle meets the primitive, to the centres of the primitive and projected circle, is equal to the angle contained by theſe circles in the ſuperficies of the ſphere; that is, to the angle contained by the planes in which theſe circles are.

PROP. VI. PROB. *Fig.* 9.

13. *To find the projected poles of a great circle* DGB, *projected upon the plane of the primitive* ABCD.

Let DGB meet the primitive in D, B; join DB, which is a diameter of the primitive,

tive, and draw AC, another diameter of the primitive, at right angles to DB, meeting the projected circle DGB in G; join DG, meeting the primitive in F; and from F, draw FM, a diameter of the primitive; the diameter NO, which is at right angles to FM, paſſes through O, N, the poles of DGB in the ſuperficies of the ſphere [def. 5. 1. Theod.]. If therefore DO be joined, meeting AC in P, P ſhall be the projected pole of DGB. In like manner, if DN be joined and produced, it ſhall meet AC, produced in the other pole of DGB. Which was to be found.

PROP. VII. PROB. *Fig.* 8.

14. *To divide a projected great circle* DGB *into degrees.*

Find P, its projected pole [6. of this]; divide the primitive ABCD into degrees, and from P, draw ſtraight lines to the points of diviſion in the primitive; theſe ſtraight lines ſhall divide the projected circle DGB into degrees. Thus PQ, PR, divide the projected circle DGB in S, T, as the primitive is divided in Q, R. Which was to be done.

Cor. From theſe two laſt propoſitions, it is manifeſt how the projected poles of a leſſer circle are found, and how a projected leſſer circle is divided into degrees: for the pole of a leſſer circle, is the pole of a great circle which is parallel to it; and the circles that divide the one into degrees, divide alſo the other into degrees.

Orthographic Projection of the Sphere.

15. The orthographic projection of the ſphere differs in nothing from the ſtereographic, except that in the orthographic projection, the eye is ſuppoſed to be at ſo great a diſtance from the ſphere, as that the rays of light proceeding from its ſeveral points to the eye may be conſidered as parallel; and for that reaſon, fig. 10. the parts of the diameter AC, repreſenting arcs of the ſemicircle ABC, are not equal to the half tangents of the arcs which they repreſent, as in prop. 1. but to the verſed ſines of theſe arcs. Thus GC, repreſenting KC [def. 9. ſect. 1. 1.], is the verſed ſine of KC; QC repreſenting PKC, is the verſed

ſine of PKC. Hence, if the ray MB paſs through the centre E, and if the rays LK, OP, meet the diameter AC in any points G, Q; EG, EQ, repreſenting BK, BP, [def. 6. ſect. 1. 1.] are equal to the coſines of KC, PKC; QG, repreſenting PK, is equal to the difference of the coſines of KC, PKC, or to the difference of the verſed ſines of theſe arcs; and QF, repreſenting PBH, is equal to the ſum of the coſines of the arcs PKC, HA, or to the difference of the verſed ſines of the arcs PKC, HKC.

For the ſame reaſon, the diameter of the projection of a leſſer circle FRGS, parallel to the primitive, is not equal to the ſum of the half tangents of that leſſer circle's diſtance from the pole of the primitive; but it is equal to the ſum of the ſines of that diſtance.

All circles of the ſphere which are at right angles to the primitive are projected into ſtraight lines. Laſtly, all circles that are inclined to the primitive are projected into ellipſes, as is ſhewn in the following propoſition.

PROP.

PROP. VIII. THEOR. *Fig.* 11.

16. *Let* ABCD *be a ſphere, and* BNDM *any circle of the ſphere inclined to the primitive; let* EG, *the ſtraight line drawn from the eye to* E *the centre of the primitive, be at right angles to the plane of the primitive; and let* BF, DH, *rays of light proceeding from any points of the ſphere to the eye, be conſidered as parallel to* EG, *and therefore* [8. 11.] *at right angles to the plane of the primitive; the points where perpendiculars to the plane of the primitive, drawn from any number of points in the circumference of the oblique circle* BNDM, *meet the plane of the primitive, are in the circumference of an ellipſis, whoſe greater axis is equal to the diameter of the oblique circle* BNDM, *and leſſer axis equal to twice the coſine of its inclination to the primitive.*

Let MN be the common ſection of the oblique circle BNDM with the primitive, MN is a diameter of BNDM, (the ſemicircle MBN being on one ſide, and MDN on the other ſide of the primitive); let BD be the diameter of BNDM, which is at right angles to MN; and from any point O, in the circumference of BNDM, draw OQ at right angles to MN, meeting it in Q; and draw OP at right angles to the plane of the primitive, meeting it in P; join PQ, and let BF meet the plane of the primitive in K. Becauſe OQ, OP, are [28. 1. & 6. 11.] parallel to BE, BK, and not in the ſame plane with them, the plane paſſing through OQ, OP, is [15. 11.] parallel to the plane paſſing through BE, BK; therefore PQ and KE, the common ſections of theſe planes, with the plane of the primitive, are [16. 11.] parallel, and the triangle OQP is therefore ſimilar to BEK: and becauſe the rectangle MEN is [35. 3.] equal to the ſquare of BE, and the rectangle MQN to the ſquare of OQ, the rectangle MEN is to the rectangle MQN, as the ſquare of BE to the ſquare of OQ; that is, (becauſe the triangles BEK, OQP, are ſimilar, and [22. 6.]), as the ſquare of KE to the ſquare of PQ; wherefore the rectangle MEN is to the rectangle MQN, as the ſquare of KE to the ſquare of PQ: but the rectangle MEN, that is, the ſquare of BE, is [19. 1.] greater than the ſquare of KE, and the rectangle MQN, that is, the ſquare of OQ, is greater than the ſquare of PQ. And becauſe BK is at right angles to the plane of the primitive, the plane of the triangle BEK is [18. 11.] at right angles to the plane of the primitive: but BE is at right angles to MN; therefore [2. & def. 4. 11.] the plane of the triangle BEK is at right angles to the plane of the circle BNDM, and [19. 11.] KE is at right angles to MN; therefore PQ is alſo at right angles to MN: wherefore [cor. 1. 6. 2. Simſ. con.] the point P is an ellipſis, whoſe greater axis is MN, and leſſer axis twice KE or KL; and KE is [def. 6. ſect. 1. 1.] the coſine of the angle BEK, which [def. 6. 11.] is the inclination of the plane of the oblique circle BNDM to the plane of the primitive. In like manner, the point where a ſtraight line drawn from any other point in the circumference BNDM, at right angles to the plane of the primitive, meets that plane, is ſhewn to be in the ſame ellipſis. Q. E. D.

Cor. Hence KP, the part of the ellipſis which is the projection of BO, any part of the circle BNDM, is found, by finding the points where the perpendiculars drawn from B, O, to the plane of the primitive, meet that plane.

SECT.

SECT. III.

Spherical Trigonometry.

1. A Spherical triangle is a figure contained by three arcs of great circles, meeting one another in the ſuperficies of a ſphere.

2. The three planes in which theſe arcs are, meet at the centre of the ſphere; and the three arcs or ſides of the ſpherical triangle are reſpectively the meaſures of the plane angles that contain the ſolid angle at the centre of the ſphere.

Hence the three ſides of a ſpherical triangle are together leſs than the whole circumference of a great circle of the ſame ſphere; becauſe every ſolid angle is contained by plane angles, which together are leſs than four right angles [21. 11.]: but the meaſures of four right angles are four quadrants, or a whole circle; therefore the three ſides of a ſpherical triangle are together leſs than the circumference of a great circle.

Hence alſo any two ſides of a ſpherical

triangle are together greater than the third; becauſe [20. 11.], when a ſolid angle is contained by three plane angles, any two of them are together greater than the third; therefore the meaſures of any two of theſe plane angles, in the ſuperficies of the ſphere, are together greater than the meaſure of the third; that is, any two ſides of a ſpherical triangle are together greater than the third ſide. It is likewiſe manifeſt, that any ſide of a ſpherical triangle is leſs than a ſemicircle.

3. The angles of a ſpherical triangle, are the inclinations of the planes in which are the arcs that contain the ſpherical angles; and the meaſure of a ſpherical angle, is an arc of a great circle intercepted between the two circles that contain the ſpherical angle at a quadrant's diſtance from the angular point in each circle; becauſe this intercepted arc is the meaſure of the inclination of the planes in which theſe arcs are.

Hence two adjacent ſpherical angles are equal to two right angles; and two vertical ſpherical angles are equal to one another.

Hence alſo the three angles of a ſpherical triangle are together greater than two right angles.

For, let AC, AB, and BC, plate 7. fig. 1. be the three ſides of a ſpherical triangle; let E be the centre of the ſphere, and join EA, EB, EC; join likewiſe AC, AB, BC, and [2. 11.] they ſhall be in one plane. Becauſe EA is equal to EB, the angle EAB [5. & 32. 1.] is leſs than a right angle, for the ſame reaſon the angle EAC is leſs than a right angle. Through the ſtraight line BC, let a plane paſs at right angles to the ſtraight line EA; CF, and BF, the common ſections of this plane, with the planes EAC, EAB, are each of them [def. 3. 11.] at right angles to EA, and [def. 6. 11.] the angle CFB is the inclination of the planes EAC, EAB: but CA [19. 1.] is greater than CF, and BA is greater than BF; therefore [21. 1.] the angle CFB is greater than the rectilineal angle CAB. In like manner, it is ſhown, that the inclination of the planes EAC, ECB, is greater than the rectilineal angle ACB; and that the inclination of the planes EAB, ECB, is greater than the rectilineal angle ABC; wherefore the inclinations of theſe three planes, or the three ſpherical angles BAC, ACB, and CBA, are together greater than the three angles of the plane triangle ACB, that is [32. 1.], greater than two right angles.

Another property of a ſpherical triangle is commonly demonſtrated along with this, viz. "That the three angles of a ſpherical "triangle are together leſs than ſix right "angles." This amounts to no more, than that any rectilineal angle is leſs than two right angles, which requires no demonſtration.

4. Becauſe the ſides of a ſpherical triangle are reſpectively the meaſures of the plane angles that contain a ſolid angle at the centre of the ſphere, the greater angle of every ſpherical triangle is ſubtended by the greater ſide, and *e contra*. For the ſame reaſon, equilateral ſpherical triangles are equiangular, and *e contra*; and the angles at the baſe of an iſoſcles ſpherical triangle are equal to one another, and *e contra*.

5. If AC and AB, two ſides of a ſpherical triangle ACB, be together equal to a ſemicircle, ABC and ACB, the ſpherical angles at the baſe, ſhall be together equal to two right angles; becauſe AC and AB are equal to a ſemicircle only, when the plane angles CEA and BEA at the centre of the ſphere, are together equal to two right angles;

angles; that is, when the inclinations of the two planes EAC, EAB, to the third plane ECB, are together equal to two right angles, and *e contra*. Likewiſe, when two ſides AC and AB are together equal to a ſemicircle, the outward ſpherical angle ACD is equal to the inward oppoſite ſpherical angle ABC at the baſe; becauſe any two adjacent ſpherical angles ACB and ACD are [art. 3.] equal to two right angles. In like manner, if any two ſides AC and AB together be greater than a ſemicircle, the ſpherical angles ACB and ABC at the baſe ſhall be greater than two right angles, and the outward ſpherical angle ACD ſhall be leſs than the inward oppoſite ſpherical angle ABC at the baſe. But if two ſides AC and AB together be leſs than a ſemicircle, the ſpherical angles ACB and ABC at the baſe ſhall be leſs than two right angles; and then the outward ſpherical angle ACD ſhall be greater than the inward oppoſite ſpherical angle ABC at the baſe.

6. If each of the ſides AC, AB, of a ſpherical triangle ACB be a quadrant, each of the angles ABC, ACB, at the baſe [art. 5.] ſhall be a right angle: If each of the ſides AC, AB, be greater than a quadrant,

each of the angles ABC, ACB, ſhall be greater than a right angle: If each of the ſides AC, AB, be leſs than a quadrant, each of the angles ABC, ACB, ſhall be leſs than a right angle: And, in the firſt caſe, if the ſpherical angle BAC, contained by the two ſides, be a right angle, the baſe BC ſhall alſo be a quadrant: but if BAC be an obtuſe angle, the baſe BC ſhall be greater than a quadrant; and if BAC be an acute angle, the baſe BC ſhall be leſs than a quadrant.

7. If the arc AB, one of the ſides of a ſpherical triangle, be a quadrant, and if the adjacent ſpherical angle ABC be a right angle, the oppoſite ſpherical angle ACB [art. 3.] ſhall be a right angle. If AB be greater than a quadrant, the oppoſite angle ſhall be greater than a right angle; and if AB be leſs than a quadrant, the oppoſite angle ſhall be leſs than a right angle, and *e contra*.

8. If ABC, an angle of a ſpherical triangle ACB, be a right angle, and if each of the ſides BA, BC, containing that angle, be greater or leſs than a quadrant, the hypothenuſe AC [art. 3.] ſhall be leſs than a quadrant. Likewiſe, if ABC be a right

angle, and if each of the other angles BAC, BCA, be either greater or leſs than a right angle, the hypothenuſe AC ſhall be leſs than a quadrant; but if ABC be a right angle, and if one of the other angles at A and C be obtuſe, and the other acute, or if one of the ſides BA, BC, containing the right angle ABC, be greater than a quadrant, and the other leſs, the hypothenuſe AC ſhall be greater than a quadrant.

9. If CA, CB, and BA, fig. 2. each of the three ſides of a right-angled ſpherical triangle ACB, be produced till CD, CE, and BF become each of them quadrants, and if an arc of a great circle be deſcribed through the three points F, D, E, the other right-angled ſpherical triangle FAD, thus deſcribed, is called *complemental* to the ſpherical triangle ACB; becauſe the parts of the one ſpherical triangle are the complements of the parts of the other to quadrants. Thus AD, one of the ſides of the ſpherical triangle FAD, is the complement of CA, the hypothenuſe of the ſpherical triangle ACB; FA, the hypothenuſe of the ſpherical triangle FAD, is the complement of AB, one of the ſides of the ſpherical triangle ACB; BE, or [art. 3.] the angle AFD of the

ſpherical triangle FAD, is the complement of CB, the other ſide of the ſpherical triangle ACB. Laſtly, FD, the remaining ſide of the ſpherical triangle FAD, is the complement of DE, that is [art. 3.], of the ſpherical angle ACB; and [art. 3.] the vertical ſpherical angles FAD, BAC, are equal.

10. The difference between a plane and a ſpherical triangle in trigonometry is, that the geometrical quantities that compoſe a plane triangle are three ſtraight lines, and three plane angles contained by theſe ſtraight lines: but the quantities that compoſe a ſpherical triangle are three arcs of great circles, and the three angles contained by theſe arcs. But becauſe a ſpherical angle is the inclination of the planes in which are the arcs that contain the ſpherical angle, a ſpherical angle does not differ from a rectilineal angle as to the buſineſs of ſpherical trigonometry.

11. And becauſe arcs of circles are the meaſures of rectilineal angles, and that theſe arcs or meaſures are found by means of ſtraight lines, viz. ſines, tangents, &c. therefore the objects of calculation in ſpherical triangles do not differ from thoſe in

plane triangles, except that no part of a ſpherical triangle is a ſtraight line, or that the parts of a ſpherical triangle which are given, and to be found, are arcs of circles.

12. Spherical triangles are by the following propoſitions reduced to plane; ſo that the parts of the ſpherical triangle correſpond to parts of the plane.

PROP. I. PROB. *Plate 7. fig. 3.*

13. *To reduce a right-angled ſpherical triangle* ABC *to a right-angled plane triangle, ſo that when the ſine of* AB, *the hypothenuſe of the ſpherical triangle, is made the hypothenuſe of the plane triangle, the ſine of* BC, *a ſide of the ſpherical triangle, ſhall be the ſide of the plane, oppoſite to a rectilineal angle equal to the oppoſite ſpherical angle* BAC.

Let D be the centre of the ſphere, and join DA, DB, DC: from B, draw BE at right angles to AD, meeting it in E; and from E, draw EF, in the plane of the arc AC, at right angles to AD, meeting DC in F, and join BF. Becauſe AD is the common ſection of the two planes ADB, ADC, and that EB, EF, are drawn in theſe planes at right angles to AD, the angle BEF [def. 6. 11.] is the inclination of the two planes ADB, ADC, that is [art. 3.], the ſpherical angle BAC. And becauſe AD is at right angles to the two ſtraight lines EB, EF, at the point E where they meet, AD [4. 11.] is at right angles to the plane of the triangle BEF; therefore [18. 11.] the plane ADC is at right angles to the plane of the triangle BEF: but the plane BDC is, by hypotheſis, at right angles to the plane ADC; wherefore the two planes BEF, BDC, are each of them at right angles to the plane ADC; and [19. 11.] their common ſection BF is at right angles to the plane ADC: therefore each of the angles BFE, BFD, is a right angle; and BF [def. 2. ſect. 1. 1.] is the ſine of BC; and BE, which [def. 2. ſect. 1. 1.] is the ſine of the hypothenuſe AB, is the hypothenuſe of the right-angled plane triangle BEF; BF, the ſine of the arc BC, is oppoſite to the rectilineal angle BEF, which is equal to the ſpherical angle BAC. Which was to be done.

PROP.

PROP. II. THEOR. *Fig.* 3.

14. *If* EB, *the ſine of* BA, *the hypothenuſe of a right-angled ſpherical triangle* ABC, *be made radius*, BF *the ſine of* BC, *either of the ſides ſhall be the ſine of the oppoſite ſpherical angle* BAC.

Join EF. Becauſe [1. of this] EFB is a right angle, if from the centre E, with the diſtance EB, a circle be deſcribed meeting EF produced, the arc of this circle intercepted between B, and the ſtraight line EF, ſhall [art. 5. ſect. 1. 1.] be the meaſure of the angle BEF, that is [1. of this], of the ſpherical angle BAC. Q. E. D.

Cor. 1. Hence, if BA, the hypothenuſe, and BC, one of the ſides of a right-angled ſpherical triangle, be each of them given, the angle BAC, oppoſite to the given ſide, is found; becauſe its ſine is the fourth proportional to the three terms, the ſine of the hypothenuſe BA, the ſine of the ſide BC, and radius.

Example. Let the hypothenuſe BA be 64°, and the ſide BC 42°, to find the ſpherical angle BAC.

The ſolution.——(1) By the logarithmic tables. Add 9.8255109, which is the logarithmic ſine of 42° to 10.0000000, which is the logarithm of radius; and from the ſum 19.8255109, ſubtract 9.9536602, which is the logarithmic ſine of 64°, the remainder 9.8718507 is the logarithmic ſine of the ſpherical angle BAC 48°, and near 7 minutes.

(2) By the ſcale of logarithmic ſines. Extend the compaſſes on the line of logarithmic ſines from 64° to 42°, the ſame diſtance ſhall reach from 90° to the number of degrees of the ſpherical angle BAC.

(3) By the ſector. Take the lateral diſtance of the ſecond term 42° off one of the lines of ſines from the centre, and make it the parallel diſtance of the firſt term 64°, alſo upon the lines of ſines; then the parallel diſtance from 90° to 90°, or radius, ſhall reach from the centre of the ſector upon one of the lines of ſines to 48°, and 6 or 7 minutes, the ſame as by the tables. See art. 40. ſect. 3. 1.

Cor. 2. If the hypothenuſe BA, and an angle BAC, be each of them given, the ſide BC, oppoſite to the given angle BAC, is found; becauſe its ſine is the fourth proportional

portional to the three terms, radius, the ſine of the angle BAC, and the ſine of the hypothenuſe BA.

Cor. 3. If one ſide BC, and the oppoſite angle BAC, be each of them given, the hypothenuſe BA is found; becauſe its ſine is the fourth proportional to the three terms, the ſine of the angle BAC, radius, and the ſine of the ſide BC. This laſt caſe is ambiguous. The propoſition and its corollaries apply equally to the hypothenuſe BA, the other ſide AC, and the oppoſite angle ABC.

Cor. 4. And becauſe the ſine of the hypothenuſe BA, the ſine of a ſide BC, radius, and the ſine of the angle BAC, oppoſite to BC, are proportionals; the rectangle contained by radius, and the ſine of BC [16. 6.], is equal to the rectangle contained by the ſine of BA, and the ſine of the angle BAC.

PROP. III. PROB. *Fig.* 3.

15. *To reduce a right-angled ſpherical triangle* ABC *to a right-angled plane triangle, ſo that when the ſine of* AC, *one of the ſides of the ſpherical triangle, is made one of the ſides of the plane triangle, the tangent of* CB, *the other ſide of the ſpherical triangle, ſhall be the ſide of the plane, oppoſite to a rectilineal angle equal to the oppoſite ſpherical angle* BAC.

Let D be the centre of the ſphere, and join DA, DC, and DB. Becauſe CG is the ſine of the arc AC, CG [def. 2. ſect. 1. 1.] is at right angles to DA: from G, draw GH in the plane of the hypothenuſe BA, at right angles to DA, meeting DB in H, and join HC. The angle CGH is [def. 6. 11.] the inclination of the planes BDA, CDA, that is [art. 3.], the ſpherical angle BAC: and becauſe DA is at right angles to each of the ſtraight lines CG, GH, it is [4. 11.] at right angles to the plane of the triangle HGC; therefore [18. 11.] the plane ADC is at right angles to the plane of the triangle HGC: but the plane CDH is, by hypotheſis, at right angles to the plane ADC; wherefore each of the planes

HGC, CDH is at right angles to the plane ADC; therefore their common ſection CH [19. 11.] is at right angles to the plane ADC; and [def. 3. 11.] each of the angles HCD, HCG, is a right angle; therefore [def, 3. ſect. 1. 1.] CH is the tangent of the arc CB. Which was to be done.

PROP. IV. THEOR. *Fig.* 3.

16. *If* GC, *the ſine of* AC, *one of the ſides of a right-angled ſpherical triangle* ABC, *be made radius*, CH, *the tangent of the other ſide* CB, *ſhall be the tangent of the oppoſite ſpherical angle* BAC.

If from the centre G, with the diſtance GC, a circle be deſcribed meeting GH, the arc of this circle, intercepted between C, and the ſtraight line GH, ſhall [art. 5. ſect. 1. 1.] be the meaſure of the angle HGC, that is, of the ſpherical angle BAC, and CH [3. of this] ſhall be the tangent of that arc. Q. E. D.

Cor. 1. Hence, if AC and CB, two ſides of a right-angled ſpherical triangle ABC, be each of them given, the ſpherical angle BAC oppoſite to one of them is found; becauſe its tangent is the fourth proportional to the three terms, the ſine of AC, the tangent of CB, and radius.

Example. Let AC be 43°, and CB 34°, to find the angle BAC.

The ſolution.——(1) By the logarithmic tables. Add 9.8289874, which is the logarithmic tangent of 34° to 10.0000000, which is the logarithm of radius; and from the ſum 19.8289874, ſubtract 9.8337833, which is the logarithmic ſine of 43°, the remainder 9.9952041 is the logarithmic tangent of the ſpherical angle BAC, 44° and about 41 minutes.

(2) By the logarithmic ſcales. Extend the compaſſes upon the line of ſines from the firſt term 43° to 90°, or radius, the ſame diſtance ſhall reach upon the line of tangents from the ſecond term 34°, to 44° and near 41′.

(3) By the ſector. Make radius, that is, the ſine of 90°, or tangent of 45°, the parallel diſtance of 43° on the lines of ſines; then the parallel tangent of 34° ſhall reach on one of the lines of tangents from the centre of the ſector to 44° 41′, as by the former methods.

Cor. 2. If the ſide AC, and the adjacent angle BAC, be each of them given, the ſide CB, oppoſite to the angle, is found; becauſe its tangent is the fourth proportional to the three terms, radius, the tangent of the angle BAC, and the ſine of the given ſide AC. In like manner, if the angle BAC, and the oppoſite ſide CB, be each of them given, the ſide AC, adjacent to the given angle, is found; becauſe its ſine is the fourth proportional to the three terms, the tangent of the angle BAC, radius, and the tangent of CB. This laſt caſe is ambiguous. The ſame proportions apply to the two ſides AC, CB, and the other angle ABC.

Cor. 3. Becauſe the ſine of a ſide AC is to the tangent of the other ſide CB, as radius to the tangent of the angle BAC oppoſite to CB; and that radius is to the tangent of the angle BAC [def. 3. & 7. ſect. 1. 1. & 4. 6.], as the cotangent of the angle BAC to radius; the ſine of AC is to the tangent of CB, as the cotangent of the angle BAC to radius; and therefore [16. 6.] the rectangle contained by radius, and the ſine of a ſide AC, is equal to the rectangle contained by the tangent of the other ſide CB, and the cotangent of the oppoſite angle BAC.

PROP. V. PROB. *Fig.* 4.

17. *To reduce a right-angled ſpherical triangle* ABC *to a right-angled plane triangle, ſo that the tangent of* AB, *the hypothenuſe of the ſpherical triangle, and the tangent of* AC, *any ſide of the ſpherical triangle, may contain the adjacent ſpherical angle* BAC.

Let D be the centre of the ſphere: join DA, DB, DC; from A, draw AK in the plane of the hypothenuſe AB, at right angles to DA, meeting DB produced in K; draw likewiſe AL, in the plane of the arc AC, at right angles to DA, meeting DC produced in L; AK and AL [def. 3. ſect. 1. 1.] are the tangents of the arcs AB, AC, each of each: join KL. It is demonſtrated in the ſame manner as in the firſt and third propoſitions, that KL, the common ſection of the planes KAL and KDL, is at right angles to the plane LDA; therefore ALK is a right angle, and the angle contained by AK and AL is [def. 6. 11. & art. 3. of this] the ſpherical angle BAC adjacent to AC. Which was to be done.

PROP. VI. THEOR. *Fig.* 4.

18. *If the tangent of* AB, *the hypothenuſe of a right-angled ſpherical triangle* ABC, *be made radius, the tangent of* AC, *either of the ſides, ſhall be the coſine of the adjacent ſpherical angle* BAC.

Becauſe ALK [5. of this] is a right angle, if from the centre A, with the diſtance AK, a circle be deſcribed meeting AL produced, the arc of this circle intercepted between K and AL, ſhall be the meaſure of the angle KAL, and AL ſhall be equal to the coſine of the ſame angle KAL; that is [def. 6. 11. & art. 3. of this], of the ſpherical angle BAC. Q. E. D.

Cor. 1. Hence, if the hypothenuſe AB, and one ſide AC of a right-angled ſpherical triangle ABC, be each of them given, the ſpherical angle BAC, adjacent to the given ſide AC, is found; becauſe its coſine is the fourth proportional to the three terms, the tangent of the hypothenuſe AB, the tangent of the ſide AC, and radius.

Example. Let AB be 64°, AC 48°, to find the ſpherical angle BAC.

The ſolution.——(1) By the logarithmic tables. Add 10.0455626, which is the logarithmic tangent of 48° to 10.0000000, which is the logarithm of radius; and from the ſum 20.0455626, ſubtract 10.3118182, which is the logarithmic tangent of 64°, the remainder 9 7337444 is the logarithmic coſine of the ſpherical angle BAC, 32° and near 48′; wherefore the ſpherical angle BAC itſelf is 57°, and near 12′.

(2) By the logarithmic ſcales. Extend the compaſſes upon the line of tangents [art. 26. ſect. 3. 1.] from 64° to 48°; the ſame diſtance ſhall reach on the line of ſines from 90°, or radius to 32°, and 47 or 48 minutes.

(3) By the ſector. Extend the compaſſes from the centre of the ſector to 48° on one of the lines of tangents above 45°, marked *tan.* and make this a parallel diſtance from 64° to 64° upon the ſame lines; then the parallel diſtance of 45° on the ſame lines alſo ſhall [art. 33. ſect. 3. 1.] be the fourth part of the ſine of 32°, and the ſame number of minutes as before.

Cor. 2. If the hypothenuſe AB, and an angle BAC, be each of them given, the ſide AC, adjacent to the given angle, is found; becauſe its tangent is the fourth proportional to the three terms, radius, the coſine of the angle BAC, and the tangent of the hypothenuſe AB. And if an angle BAC, and the adjacent ſide AC, be each of them given, the hypothenuſe AB is found; becauſe its tangent is the fourth proportional to the three terms, the coſine of the angle BAC, radius, and the tangent of the ſide AC.

Cor. 3. Becauſe the tangent of the hypothenuſe AB is to the tangent of the ſide AC, as radius to the coſine of the angle BAC; alternately, the tangent of AB is to radius, as the tangent of AC to the coſine of the angle BAC: but the tangent of AB is to radius [def. 3. & 7. ſect. 1. 1. & 4. 6.], as radius to the cotangent of AB; therefore radius is to the cotangent of AB, as the tangent of AC to the coſine of the angle BAC, and [16. 6.] the rectangle contained by radius and the coſine of the angle BAC, is equal to the rectangle contained by the cotangent of AB, and the tangent of AC.

PROP.

PROP. VII. THEOR. *Fig.* 2.

19. *If the coſine of* CA, *the hypothenuſe of a right-angled ſpherical triangle* ACB, *be made radius, the cotangent of one of the adjacent angles* BCA *ſhall be the tangent of the other adjacent angle* BAC.

Let FAD be the complemental ſpherical triangle to ACB. Becauſe FAD is a ſpherical triangle right-angled at D, if the ſine of one of its ſides AD be made radius, the tangent of the other ſide DF [4. of this] ſhall be the tangent of the oppoſite ſpherical angle DAF or BAC: but AD [art. 9.] is the complement of CA, and FD is the complement of the ſpherical angle BCA; wherefore, if the coſine of CA be made radius, the cotangent of the angle BCA ſhall be the tangent of the angle BAC. Q. E. D.

Cor. 1. Hence, if the hypothenuſe CA, and one angle BCA of a right-angled ſpherical triangle ACB, be each of them given, the other angle BAC is found; becauſe its tangent is the fourth proportional to the three terms, the coſine of CA, the cotangent of the angle BCA, and radius.

Example. Let the hypothenuſe CA be 42°, and the angle BCA 58°, to find the angle BAC.

The ſolution. —— (1) By the logarithmic tables. Add 9.7957892, which is the logarithmic cotangent of the angle BCA, to 10.0000000, which is the logarithm of radius; and from the ſum 19.7957892, ſubtract 9.8710735, which is the logarithmic coſine of CA; the remainder 9,9247157 is the logarithmic tangent of the angle BAC 40° 3′.

(2) By the logarithmic ſcales. Extend the compaſſes from the ſine of the complement of CA to radius; the ſame diſtance ſhall reach from the tangent of the complement of the angle BCA to the tangent of the angle BAC.

(3) By the ſector. Make the lateral diſtance of the cotangent of the angle BCA the parallel diſtance of the coſine of CA; then the parallel radius ſhall be the tangent of the angle BAC.

Cor. 2. Likewiſe, if the hypothenuſe CA, and the angle BAC, be each of them given, the angle BCA is found; becauſe its cotangent is the fourth proportional to the three terms, radius, the tangent of BAC,

and

and the coſine of CA, or becauſe its tangent is the fourth proportional to the three terms, the coſine of CA, the cotangent of BAC, and radius. If the two angles BCA and BAC, adjacent to the hypothenuſe CA, be each of them given, the hypothenuſe is found; becauſe its coſine is the fourth proportional to the three terms, the tangent of the angle BAC, radius, and the cotangent of the other angle BCA.

Cor. 3. And becauſe the coſine of the hypothenuſe CA, the cotangent of the angle BCA, radius, and the tangent of the other angle BAC, are proportionals; and that the cotangent of any angle BAC is to radius [def. 3. & 7. ſect. 1. 1. & 4. 6.], as radius to the tangent of the ſame angle; the coſine of the hypothenuſe CA, the cotangent of the angle BCA, and radius, are proportionals; therefore [16. 6.] the rectangle contained by radius, and the coſine of the hypothenuſe CA, is equal to the rectangle contained by the cotangents of the adjacent angles BCA, BAC.

PROP. VIII. THEOR. *Fig.* 2.

20. *If the coſine of* BA, *one of the ſides of a right-angled ſpherical triangle* ACB, *be made radius, the coſine of the hypothenuſe* CA *ſhall be the coſine of the other ſide* CB.

Let FAD [art. 9.] be the complemental ſpherical triangle to ACB. Becauſe FAD is a ſpherical triangle right-angled at D, if the ſine of the hypothenuſe AF be made radius, the ſine of the ſide AD ſhall be [2. of this] the ſine of the oppoſite ſpherical angle AFD: but AF [art. 9.] is the complement of BA, AD is the complement of CA, and the ſpherical angle AFD is the complement of CB; wherefore, if the coſine of BA be made radius, the coſine of CA ſhall be the coſine of CB. Q. E. D.

Cor. 1. Hence if one ſide BA, and the hypothenuſe CA, of a right-angled ſpherical triangle ACB, be each of them given, the other ſide CB is found; becauſe its coſine is the fourth proportional to the three terms, the coſine of BA, the coſine of CA, and radius.

Example. Let BA be 30° 40′, and CA 52° 30′, to find CB.

The ſolution.——(1) By the logarithmic tables. Add 9.7844471, which is the logarithmic coſine of 52° 30′ to 10.0000000, which is the logarithm of radius; and from the ſum 19.7844471, ſubtract 9.9345738, the remainder 9.8498733 is the logarithmic coſine of CB 45° 3′; wherefore CB is 44° 57′.

(2) By the logarithmic ſcales. Extend the compaſſes upon the line of ſines from 59° 20′, which is the complement of BA, to 37° 30′; the ſame diſtance ſhall reach from 90° to the number of degrees which is the complement of CB.

(3) By the ſector. Make the lateral diſtance of the ſine of 37° 30′ the parallel diſtance of the ſine of 59° 20′; then the parallel radius, or 90°, ſhall reach from the centre to the ſine of the complement of CB.

Cor. 2. Likewiſe if BA and BC, the two ſides containing the right angle of a ſpherical triangle, be each of them given, the hypothenuſe CA is found; becauſe its coſine is the fourth proportional to the three terms, radius, the coſine of BC, and the coſine of BA: and if the hypothenuſe CA, and the ſide BC, be each of them given, the ſide BA is found in the ſame manner that CB was found.

Cor. 3. And becauſe the coſine of a ſide BA, the coſine of the hypothenuſe CA, radius, and the coſine of the other ſide CB, are proportionals, the rectangle contained by radius, and the coſine of the hypothenuſe CA [16. 6.], is equal to the rectangle contained by the coſines of the ſides BA, BC.

PROP. IX. THEOR. *Fig.* 2.

21. *If the coſine of* BA, *one of the ſides of a right-angled ſpherical triangle* ACB, *be made radius, the coſine of the oppoſite ſpherical angle* BCA, *ſhall be the ſine of the other acute ſpherical angle* BAC, *which is adjacent to* BA.

Let FAD be the complemental ſpherical triangle to ACB. Becauſe FAD is a ſpherical triangle right-angled at D, if the ſine of the hypothenuſe AF be made radius, the ſine of the ſide FD ſhall [2. of this] be the ſine of the oppoſite ſpherical angle FAD, or

BAC [art. 3.]: but AF [art. 9.] is the complement of BA, FD is the complement of the ſpherical angle BCA; wherefore, if the coſine of BA be made radius, the coſine of the oppoſite ſpherical angle BCA ſhall be the ſine of the other acute ſpherical angle BAC. Q. E. D.

Cor. 1. Hence, if one ſide BA, and the oppoſite ſpherical angle BCA of a right-angled ſpherical triangle ACB, be each of them given, the other angle BAC is found; becauſe its ſine is the fourth proportional to the three terms, the coſine of of BA, the coſine of the angle BCA, and radius.

Example. Let BA be 30° 40′, and the angle BCA 38°, to find the angle BAC.

The ſolution.——(1) By the logarithmic tables. Add 9.8905026, which is the logarithmic coſine of 38°, to 10.0000000, which is the logarithm of radius; and from the ſum 19.8905026, ſubtract 9.9353204, the remainder 9.9551822 is the logarithmic ſine of the angle BAC 64° 24′.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of ſines from 59° 30′ to 51°; the ſame diſtance ſhall reach

from 90° to the number of degrees and minutes of the angle BAC.

(3) By the ſector. Make the lateral diſtance of the ſine of 51° the parallel diſtance of the ſine of 59° 30′; the parallel radius, or 90°, ſhall reach from the centre to the ſine of BAC. This caſe is ambiguous.

Cor. 2. If one ſide BA, and the adjacent acute angle BAC of a right-angled ſpherical triangle ACB, be each of them given, the oppoſite angle BCA is found; becauſe its coſine is the fourth proportional to the three terms, radius, the ſine of BAC, and the coſine of BA. In like manner, if two angles BAC, and BCA of a right-angled ſpherical triangle ACB, be each of them given, either of the ſides BA is found; becauſe its coſine is the fourth proportional to the three terms, the ſine of the adjacent angle BAC, radius, and the coſine of the oppoſite angle BCA.

Cor. 3. And becauſe the coſine of the ſide BA, the coſine of the oppoſite angle BCA, radius, and the ſine of the adjacent angle BAC, are proportionals, the rectangle contained by radius, and the coſine of the angle BCA [16. 6.], is equal to the rectangle

angle contained by the coſine of BA and the ſine of BAC.

Of Napier's rules for ſolving the caſes of right-angled ſpherical triangles.

22. By an accurate conſideration of the proportions by which the caſes of right-angled ſpherical triangles are ſolved, the inventor of the logarithms invented alſo two ſhort rules for ſolving the caſes of right-angled ſpherical triangles, comprehending all the caſes that are deduced from the foregoing propoſitions. See *Mirif. logar. canon. deſcriptio, lib.* 2. *cap.* 4.

He obſerved, that three of the five circular parts, viz. the complement of the hypothenuſe CA, fig. 2. the complement of each of the angles BAC, BCA, and each of the ſides BA, BC, or the equivalents of theſe five, occur in every caſe. Any one of theſe three parts he calls the *intermediate* or *middle part*, and the other two he calls the *extremes*. Thus, when the complement of the hypothenuſe CA is the middle part, the complements of the angles BAC, BCA, are the extremes; ſo likewiſe are the ſides BA, BC, and ſo on.

The extremes again he divides into ſuch as are contiguous or adjacent, and ſuch as are oppoſite, *aut circumponuntur aut opponuntur*. For example, when the complement of the hypothenuſe CA is the middle part, the complements of the angles BAC, BCA, are adjacent extremes: but the ſides BA, BC, are oppoſite extremes; for the complements of the angles BAC, BCA, are conſidered as contiguous to the complement of the hypothenuſe CA, and as it were ſeparating it from the ſides; for which reaſon the ſides are in this caſe called *oppoſite extremes*. In like manner, when the complement of the angle BAC is the middle part, the complement of the hypothenuſe CA, and the ſide BA, are adjacent extremes; but the complement of the angle BCA, and the ſide BC, are oppoſite extremes. So likewiſe, when the ſide BA is the middle part, the complement of the angle BAC, and the ſide BC, are adjacent extremes; but the complement of the hypothenuſe CA, and the complement of the angle BCA, are oppoſite extremes. The adjacent and oppoſite extremes are diſtinguiſhed in the ſame manner, when the complement of the angle BCA, and the ſide BC, are middle parts.

RULE I.

23. The rectangle contained by radius, and the ſine of the middle part, is equal to the rectangle contained by the coſines of the oppoſite extremes. Fig. 2.

(1) When the complement of the hypothenuſe CA is the middle part, the ſides BA and BC are [art. 22.] oppoſite extremes; and [cor. 3. 8. of this] the rectangle contained by radius and the coſine of CA is equal to the rectangle contained by the coſines of the ſides BA, BC.

(2) When the complement of an angle BCA is the middle part, the complement of the angle BAC, and the ſide BA, are oppoſite extremes; and [cor. 3. 9. of this] the rectangle contained by radius and the coſine of BCA, is equal to the rectangle contained by the coſine of BA, and the ſine of BAC, which is the coſine of the complement of BAC.

(3) When a ſide BC is the middle part, the complement of the hypothenuſe CA,

and the complement of the angle BAC, are oppoſite extremes; and [cor. 4. 2. of this] the rectangle contained by radius and the ſine of a ſide BC, is equal to the rectangle contained by the ſine of the hypothenuſe CA, and the ſine of the angle BAC, oppoſite to the ſide BC; but the hypothenuſe CA, and the angle BAC, are the complements of their complements. In like manner, the rule is demonſtrated, when the complement of the angle BAC, and the ſide BA, are middle parts.

Hence [14. 6.] a ſide of the firſt rectangle is to a ſide of the ſecond, as the remaining ſide of the ſecond to the remaining ſide of the firſt.

RULE II.

24. The rectangle contained by radius and the ſine of the middle part, is equal to the rectangle contained by the tangents of the adjacent extremes. Fig. 2.

(1) When the complement of the hypothenuſe CA is the middle part, the complements of the angles BAC and BCA are [art. 22.] the adjacent extremes; and [cor.

3. 7. of this] the rectangle contained by radius, and the coſine of the hypothenuſe CA, is equal to the rectangle contained by the cotangents of the angles BAC, BCA.

(2) When the complement of an angle BCA is the middle part, the complement of the hypothenuſe CA, and the ſide BC, are the adjacent extremes; and [cor. 3. 6. of this] the rectangle contained by radius, and the coſine of the angle BCA, is equal to the rectangle contained by the cotangent of the hypothenuſe CA and the tangent of the ſide BC.

(3) When a ſide BC is the middle part, the complement of the angle BCA, and the other ſide BA, are the adjacent extremes; and [cor. 3. 4. of this] the rectangle contained by radius, and the ſine of the ſide BC, is equal to the rectangle contained by the cotangent of the angle BCA, and the tangent of the other ſide BA. In like manner, the rule is demonſtrated, when the complement of the angle BAC, and the ſide BA, are middle parts.

Hence [14. 6.] a ſide of the firſt rectangle is to a ſide of the ſecond, as the remaining ſide of the ſecond to the remaining ſide of the firſt.

Of the number of caſes in the ſolutions of right-angled ſpherical triangles.

25. The parts of a right-angled ſpherical triangle ACB, fig. 2. that are conſidered in trigonometry, are five in number, as in a right-angled plane triangle, viz. the hypothenuſe, two ſides, and two angles: and two of theſe five parts muſt be given in order to find the third; becauſe radius is one of the terms of the four proportionals by which any caſe is ſolved. Likewiſe the number of poſſible combinations of three parts of the five, and of two parts of theſe three parts with the third, is the ſame as in a right-angled plane triangle [art. 7. ſect. 4. 1.]. There are more caſes, however, in the ſolutions of right-angled ſpherical triangles, than in right-angled plane triangles: for becauſe the three angles of a ſpherical triangle are together [art. 3. of this] greater than two right angles, each angle of a right-angled ſpherical triangle muſt be found ſeparately. All the general caſes of right-angled ſpherical triangles that are deduced from the foregoing propoſitions, viz. the 2d, 4th, 6th, 7th, 8th, and 9th, are eighteen; namely, three caſes from each of

theſe propoſitions. Two of theſe eighteen caſes are unneceſſary, viz. the caſe in cor. 1. 7. is the ſame general caſe with the firſt caſe in cor. 2. 7.; becauſe, combining the hypothenuſe CA with one angle BCA, in order to find the other angle BAC, is the ſame general caſe with combining the hypothenuſe CA with this laſt angle in order to find the firſt: and the caſe in cor. 1. 8. is the ſame general caſe with the laſt caſe in cor. 2. 8.; becauſe, combining the hypothenuſe CA with one ſide BA, in order to find the other ſide BC, is the ſame general caſe with combining the hypothenuſe CA with this laſt ſide, in order to find the firſt. The caſes of right-angled ſpherical triangles are therefore ſixteen. The ſame number of caſes are deduced from Lord Napier's two rules; which rules are commonly uſed in practice by ſuch as are acquainted with them, as they are eaſily remembered. The ſixteen caſes of right-angled ſpherical triangles are ſet down in the following table; and in the column of the proportionals, references are made to Napier's rules, as well as to the propoſitions.

In ſolving the caſes of right-angled ſpherical triangles by the rules, the learner will obſerve, that the given ſide of the rectangle whoſe ſide is ſought, is the firſt term of the proportionals; and that either ſide of the other rectangle [16. 5.] may be the ſecond term.

Solutions of the caſes of right-angled ſpherical triangles. Plate 7. fig. 2.

	Given.	Sought.	Proportionals.
1.	The hypoth. CA, and a ſide BA.	The oppoſite angle BCA.	The ſine of CA is to the ſine of BA, as radius to the ſine of BCA [cor. 1. 2. of this, or rule 1. BA being middle part.]
2.	The hypoth. CA, and an angle BCA.	The oppoſite ſide BA.	Radius is to the ſine of BCA, as the ſine of CA to the ſine of BA, [cor. 2. 2. of this, or rule 1. BA being middle part.]
3.	A ſide BA, and the oppoſite angle BCA.	The hyp. CA.	The ſine of the angle BCA is to radius, as the ſine of BA to the ſine of CA, [cor. 3. 2. of this, or rule 1. BA being middle part], This caſe is ambiguous.
4.	Two ſides BC, BA.	An angle BCA.	The ſine of BC is to the tangent of BA, as radius to the tangent of BCA, [cor. 1. 4. of this, or rule 2. BC being middle part.]
5.	A ſide BC, and the adjacent angle BCA.	The other ſide BA.	Radius is to the tangent of BCA, as the ſine of CB to the tangent of BA, [cor. 2. 4. of this, or rule 2. BC being middle part.]
6.	An angle BCA, and the oppoſite ſide BA.	The other ſide BC	The tangent of BCA is to radius, as the tangent of BA to the tangent of BC, [cor. 2. 4. of this, or rule 2. BC being middle part]. This caſe is ambiguous.
7.	The hypoth. CA, and a ſide BC.	The adjacent angle BCA.	The tangent of CA is to the tangent of BC, as radius to the coſine of BCA, [cor. 1. 6. of this, or rule 2. the complement of BCA being middle part.]

	Given.	Sought.	Proportionals.
8.	The hypoth. CA, and an angle BCA.	The adjacent ſide BC.	Radius is to the coſine of BCA, as the tangent of CA to the tangent of BC, [cor. 2. 6. of this, or rule 2. the complement of BCA being middle part.]
9.	An angle BCA, and the adjacent ſide BC.	The hyp. CA.	The coſine of BCA is to radius, as the tangent of BC to the tangent of CA, [cor. 2. 6. of this, or rule 2. the complement of BCA being middle part.]
10	The hypoth. CA, and an angle BCA.	The other angle BAC	The coſine of CA is to the cotangent of BCA, as radius to the tangent of BAC, [cor. 1. 7. of this, or rule 2. the complement of CA being middle part.]
11	The two angles BCA, BAC.	The hyp. CA.	The tangent of BAC is to radius, as the cotangent of BCA to the coſine of CA, [cor. 2. 7. of this, or rule 2. the complement of CA being middle part.]
12	A ſide BA, and the hyp. CA.	The other ſide BC.	The coſine of BA is to the coſine of CA, as radius to the coſine of BC, [cor. 1. 8. of this, or rule 1. the complement of CA being middle part.]
13	Two ſides BA, BC.	The hyp. CA.	Radius is to the coſine of BC, as the coſine of BA to the coſine of CA, [cor. 2. 8. of this, or rule 1. the complement of CA being middle part.]
14	A ſide BA, and the oppoſite angle BCA.	The other angle BAC	The coſine of BA is to the coſine of BCA, as radius to the ſine of BAC, [cor. 1. 9. of this, or rule 1. the complement of BCA being middle part]. This caſe is ambiguous.
15	A ſide BA, and the adjacent angle BAC.	The other angle BCA	Radius is to the ſine of BAC, as the coſine of BA to the coſine of BCA, [cor. 2. 9. of this, or rule 1. the complement of BCA being middle part.]
16	The two angles BAC, BCA.	A ſide BA.	The ſine of BAC is to radius, as the coſine of BCA to the coſine of BA, [cor. 2. 9. of this, or rule 1. the complement of the angle BCA being middle part.]

Of oblique-angled ſpherical triangles. Fig. 5.

26. If each of the angles ACB, ABC, which are at the baſe of an oblique-angled ſpherical triangle BAC, be either obtuſe or acute, the arc of a great circle AN, deſcribed through the other angular point A, at right angles to the baſe, ſhall be within the triangle BAC. But if one of the angles ACD, at the baſe of the triangle CAD, be acute, and the other angle ADC at the baſe obtuſe, the perpendicular AN ſhall be without the triangle CAD [art. 5.].

27. If each of the angles ADB, DBA, and DAB, of a ſpherical triangle BAD, be acute, each of the ſides AB, AD, and DB, ſhall be leſs than a quadrant, and *e contra* [art. 6.]: and therefore, if one ſide of an oblique-angled ſpherical triangle be greater than a quadrant, the oppoſite angle ſhall be obtuſe; if each of two angles be obtuſe, and the third acute, each of the ſides oppoſite to the obtuſe angles ſhall be greater than a quadrant, and the ſide oppoſite to the acute angle ſhall be leſs than a quadrant.

28. If each of the ſides AC, AB, and

CB, of a ſpherical triangle BAC, be greater than a quadrant; or if each of two ſides be greater than a quadrant, and if the third ſide be a quadrant, each of the angles ſhall be obtuſe [art. 5. & 6.]: and if each of two ſides of an oblique-angled ſpherical triangle be leſs than a quadrant, but the third ſide greater than a quadrant, the angle oppoſite to the greater ſide ſhall be obtuſe, and each of the other angles ſhall be acute.

PROP. X. THEOR. *Fig.* 5.

29. *The ſines of any two ſides* AC, AB, *of any ſpherical triangle* BAC, *are to one another, as the ſines of the angles* ABC, ACB, *oppoſite to theſe ſides; that is, the ſine of* AC *is to the ſine of* AB, *as the ſine of the angle* ABC *to the ſine of the angle* ACB.

Through A, deſcribe an arc of a great circle at right angles to the baſe CB, meeting it in N. Becauſe [cor. 1. 2. of this] the ſine of AC is to the ſine of AN, as radius to the ſine of the angle ACN, the rectangle contained by the ſine of AC, and the ſine of the angle ACN [16. 6.], is equal to the rectangle contained by radius and the ſine of AN. For the ſame reaſon, the rectangle contained by the ſine of AB, and the ſine of the angle ABN, is equal to the rectangle contained by radius and the ſine of AN; therefore, the rectangle contained by the ſine of AC, and the ſine of the angle ACN, is equal to the rectangle contained by the ſine of AB, and the ſine of the angle ABN; therefore [14. 6.] the ſine of AC is to the ſine of AB, as the ſine of ABC to the ſine of ACB. Q. E. D.

The propoſition is manifeſt likewiſe when the perpendicular arc AN falls without the triangle DAC; becauſe the arc AD is equal to the arc AB, and the angle ABD to the angle ADB; and likewiſe, that the ſine of the angle ADC [art. 3. of this, and def. 2. ſect. 1. 1.] is equal to the ſine of ADB.

Cor. 1. Hence, if two ſides AC, AB, and an angle ABC oppoſite to one of them, be each of them given, the angle ACB oppoſite to the other is found; becauſe its ſine is the fourth proportional to the three terms, the ſine of AC, the ſine of AB, and the ſine of the other angle ABC.

Example. Let AC be 54°, AB 38°, and the angle ABC 47°, to find the angle ACB.

The

The ſolution. —— (1) By the logarithmic tables. Add 9.7893420, which is the logarithmic ſine of AB 38°, to 9.8641275, which is the logarithmic ſine of the angle ABC 47°; and from the ſum 19.6534695, ſubtract 9.9079576, which is the logarithmic ſine of AC 54°; the remainder 9.7455119 is the logarithmic ſine of the angle ACB, 33° and between 49 and 50 minutes.

(2) By the logarithmic ſcale. Extend the compaſſes on the line of ſines from 54° to 38°; the ſame diſtance ſhall reach from 47° to the number of degrees and minutes of the angle ACB.

(3) By the ſector. Make the lateral ſine of the ſecond term 38°, the parallel ſine of the firſt term 54°; then the parallel ſine of the third term 47° ſhall be the lateral ſine of the fourth term, viz. the ſine of the angle ACB. The angle ABC is found the ſame way, if the ſides and the angle ACB be each of them given.

Cor. 2. If two angles ABC, ACB, and a ſide AC, be each of them given, the other ſide AB is found; becauſe its ſine is the fourth proportional to the three terms, the ſine of the angle ABC, the ſine of the angle ACB, and the ſine of the ſide AC. In like manner, if each of the angles ABC, ACB, and a ſide AB, be given, the other ſide AC is found.

PROP. XI. THEOR. *Fig.* 5.

30. *The ſame conſtruction being made as in the preceding propoſition, the coſines of the ſides* AC, AB, *are to one another as the coſines of the arcs* CN, BN, *into which the baſe is divided by the perpendicular arc* AN; *that is, the coſine of* AC *is to the coſine of* AB, *as the coſine of* CN *to the coſine of* BN.

Becauſe [cor. 1. 8. of this] the coſine of AN is to the coſine of AC, as radius to the coſine of CN; alternately, the coſine of AN is to radius, as the coſine of AC to the coſine of CN. For the ſame reaſon, the coſine of AN is to radius, as the coſine of AB to the coſine of BN; therefore [11. 5.] the coſine of AC is to the coſine of CN, as the coſine of AB to the coſine of BN; alternately, the coſine of AC is to the coſine of AB, as the coſine of CN to the coſine of BN. Q. E. D.

Cor. 1. Let AC be 54°, AB 38°, and the arc CN 52°, to find the other arc BN.

The ſolution.—— (1) By the logarithmic tables. Add 9.7893420, which is the logarithmic coſine of CN 52°, to 9.8965321, which is the logarithmic coſine of AB 38°; and from the ſum 19.6858741, ſubtract 9.7692187; the remainder 9.9166554 is the logarithmic coſine of BN, 34° and about 23 minutes.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of ſines from 36°, which is the complement of the firſt term AC, to 52°, which is the complement of the ſecond term AB; the ſame diſtance ſhall reach from 38°, which is the complement of the third term CN, to 55° 37′, viz. the complement of BN.

(3) By the ſector. Make the lateral ſine of 52° the parallel ſine of 36°; then the parallel ſine of 38° ſhall be the lateral coſine of CN.

Cor. 2. If the two arcs CN, BN, into which the baſe BC is divided by the perpendicular AN, and a ſide AC, be each of them given, the other ſide AB is found; becauſe its coſine is the fourth proportional to the three terms, the coſine of CN, the coſine of BN, and the coſine of AC.

PROP. XII. THEOR. *Fig.* 5.

31. *The coſines of the angles* ACB, ABC, *at the baſe of any ſpherical triangle, are to one another as the ſines of the angles* CAN, BAN, *into which the angle* BAC, *at the vertex, is divided by the perpendicular arc* AN; *that is, the coſine of* ACB *is to the coſine of* ABC, *as the ſine of* CAN *to the ſine of* BAN.

Becauſe [cor. 1. 9. of this] the coſine of AN is to the coſine of the angle ACN, as radius to the ſine of the angle CAN; alternately, the coſine of AN is to radius, as the coſine of ACN to the ſine of CAN. For the ſame reaſon, the coſine of AN is to radius, as the coſine of ABN to the ſine of BAN; therefore [11. 5.] the coſine of ACN is to the ſine of CAN, as the coſine of ABN to the ſine of BAN; alternately, the coſine of ACN is to the coſine of ABN, as the ſine of CAN to the ſine of BAN. Q. E. D.

Cor. 1. Let the angle ACB be 42°, the

angle ABC 49°, and the angle CAN 54°, to find BAN, the remaining part of the angle BAC.

The ſolution.—— (1) By the logarithmic tables. Add 9.8169429, which is the logarithmic coſine of the angle ABC 49°, to 9.9079576, which is the logarithmic ſine of CAN 54°; and from the ſum 19.7249005 ſubtract 9.8710735, which is the logarithmic coſine of ACB 42°; the remainder 9.8538270 is the logarithmic ſine of the angle BAN 45° 35′.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of ſines from the coſine of ACB to the coſine of ABC; the ſame diſtance ſhall reach from the ſine of CAN to the ſine of BAN.

(3) By the ſector. Make the lateral coſine of ABC the parallel coſine of ACB; then the parallel ſine of CAN ſhall be the lateral ſine of BAN.

Cor. 2. If each of the angles CAN, BAN, into which the angle at the vertex is divided by the perpendicular AN, and ACB, one of the angles at the baſe, be given, the other angle ABC at the baſe is found; becauſe its coſine is the fourth proportional to the three terms, the ſine of

CAN, the ſine of BAN, and the coſine of ACB.

PROP. XIII. THEOR. *Fig.* 5.

32. *The tangents of the angles* ACB, ABC, *at the baſe of any ſpherical triangle, are to one another reciprocally, as the ſines of the arcs* CN, BN, *adjacent to theſe angles into which the baſe is divided by the perpendicular arc* AN; *that is, the tangent of the angle* ACB *is to the tangent of the angle* ABC, *as the ſine of* BN *to the ſine of* CN.

Becauſe [cor. 1. 4. of this] the ſine of CN is to the tangent of AN, as radius to the tangent of ACN, the rectangle contained by radius, and the tangent of AN [16. 6.], is equal to the rectangle contained by the ſine of CN, and the tangent of ACN. For the ſame reaſon, the rectangle contained by radius, and the tangent of AN, is equal to the rectangle contained by the ſine of BN, and the tangent of ABN; therefore the rectangle contained by the ſine of CN, and the tangent of ACN, is equal to the rectangle contained by the ſine of BN, and the tangent of ABN; and therefore [14. 6.] the tangent of ACN is to the tangent of ABN, as the ſine of BN to the ſine of CN. Q. E. D.

Cor. 1. Let the angle ACB be 42°, the angle ABC 49°, and the arc BN 34° 23′, to find the arc CN.

The ſolution.—— (1) By the logarithmic tables. Add 9.7518385, which is the logarithmic ſine of BN 34° 23′, to 10.0608369, which is the logarithmic tangent of the angle ABC 49°; and from the ſum 19.8126754 ſubtract 9.9544374, which is the logarithmic tangent of the angle ACB 42°; the remainder 9.8582380 is the logarithmic ſine of the arc CN, 46° and near 11 minutes.

(2) By the logarithmic ſcales. Becauſe the angle ACB 42° is leſs than 45°, and that the angle ABC 49° is greater than 45°, the diſtance between 42° and 49°, upon the ſcale of logarithmic tangents [art. 26. ſect. 3. 1.], is equal to the diſtance between 42° and 49°, together with twice the diſtance between 42° and 45°; therefore extend the compaſſes from 45° to 49°; and obſerve the point to which this diſtance reaches from 42° towards the beginning of the ſcale; the diſtance between this point

and 45° ſhall reach upon the line of ſines from 34° 23′ to the number of degrees and minutes ſought, 46° 11′. See cor. 1. 5. ſect. 4. 1.

(3) By the ſector. Take the tangent of 49° from the centre of the ſector upon one of the lines of tangents above 45°, and make it the parallel tangent of 42°; then the parallel ſine of 34° 23′ ſhall be the fourth part of the ſine of the number of degrees and minutes ſought [art. 33. ſect. 3. 1.].

Cor. 2. If each of the arcs CN, BN, and ABC, one of the angles at the baſe be given, the other angle ACB at the baſe is found; becauſe its tangent is the fourth proportional to the three terms, the ſine of CN, the ſine of BN, and the tangent of ABC.

PROP. XIV. THEOR. *Fig.* 5.

33. *The tangents of the ſides* AC, AB, *of any ſpherical triangle, are to one another reciprocally, as the coſines of the angles* CAN, BAN, *adjacent to theſe ſides into which the angle* BAC *is divided by the perpendicular arc* AN; *that is, the tangent of* AC *is to the tangent of* AB, *as the coſine of the angle* BAN *to the coſine of the angle* CAN.

Becauſe [cor. 1. 6. of this] the tangent of AC is to the tangent of AN, as radius to the coſine of the angle CAN, the rectangle contained by radius, and the tangent of AN [16. 6.], is equal to the rectangle contained by the tangent of AC, and the coſine of CAN. For the ſame reaſon, the rectangle contained by radius, and the tangent of AN, is equal to the rectangle contained by the tangent of AB, and the coſine of BAN; therefore the rectangle contained by the tangent of AC, and the coſine of CAN, is equal to the rectangle contained by the tangent of AB, and the coſine of BAN; and therefore [14. 6.] the tangent of AC is to the tangent of AB, as the coſine of BAN to the coſine of CAN. Q. E. D.

Cor. 1. Let AC be 62°, AB 48°, and the angle BAN 37°, to find the angle CAN.

The ſolution. —— (1) By the logarithmic tables. Add 9.9023486, which is the logarithmic coſine of the angle BAN, to 10.0455626, which is the logarithmic tangent of AB; and from the ſum 19.9479112 ſubtract 10.2743256, which is the logarithmic tangent of AC; the remainder 9.6735856 is the logarithmic coſine of CAN, which is therefore 61°, and about 52 minutes.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of tangents from the tangent of 62° to the tangent of 48°; the ſame diſtance ſhall reach on the line of ſines from the coſine of BAN to the coſine of CAN.

(3) By the ſector. Make the lateral coſine of the angle BAN 37°, viz. the ſine of 53° the parallel tangent of 62°; then the parallel tangent of 48° ſhall be the lateral coſine of CAN.

Cor. 2. If each of the angles CAN, BAN, into which the angle at the vertex

BAC is divided by the perpendicular arc AN, and one ſide AB, be given, the other ſide AC is found; becauſe its tangent is the fourth proportional to the three terms, the coſine of CAN, the coſine of BAN, and the tangent of AB.

PROP. XV. THEOR. *Fig.* 5.

34. *The tangent of half the baſe* BC, *of any ſpherical triangle* BAC, *is to the tangent of half the ſum of the two ſides* AC *and* AB, *as the tangent of half the difference of the ſides to the tangent of half the difference of the arcs* CN, BN, *into which the baſe is divided by the arc* AN, *which is deſcribed through the angular point* A, *at right angles to the baſe* BC.

Let CG be the common ſection of the planes in which are the arcs CA, CB; CG is [11. 1. Theod.] a diameter of the ſphere. From the pole A, with the diſtance AB, the leſſer of the two ſides, deſcribe a circle meeting CB in D, and CA in F, E; CE is equal to the ſum of the two ſides AC, AB, and CF is their difference. Likewiſe, becauſe A is the pole of the circle

cle DBE, AB [def. 5. 1. Theod. & 28. 3.] is equal to AD; therefore [art. 4. of this] BN is equal to DN, and CD is the difference of CN, BN. Let a plane touch the ſphere in the point C; and from G, draw a ſtraight line through each of the points E, B, D, F, meeting this plane in K, H, L, M; the three points C, L, H, are [art. 6. per.] in a ſtraight line; ſo are likewiſe the three points C, M, K: and becauſe CG [4. 1. Theod.] is at right angles to the plane touching the ſphere in C, each of the angles GCH, GCK [def. 3. 11.], is a right angle; wherefore, if G be made the centre, and GC radius, CH ſhall be the tangent of the angle CGH, CK ſhall be the tangent of the angle CGK, CM ſhall be the tangent of the angle CGM, and CL ſhall be the tangent of the angle CGL. But theſe angles are each of them [20. 3.] the halves of the angles at the centre of the ſphere, whoſe meaſures are the arcs CB, CE, CF, CD; therefore CH is the tangent of the half of CB, CK is the tangent of the half of CE, CM is the tangent of the half of CF, and CL is the tangent of the half of CD. And becauſe the points B, E, F, D, are in the circumference of a circle which is in the ſuperficies of the ſphere, the points H, K, M, L, are [cor. 2. 4. ſect. 2. 2.] in the circumference of a circle which is in the plane that touches the ſphere in the point C; therefore [cor. 36. 3.] the rectangle HCL is equal to the rectangle KCM; and [14. 6.] CH is to CK, as CM to CL; that is, the tangent of half the baſe is to the tangent of half the ſum of the two ſides, as the tangent of half the difference of the ſides to the tangent of half the difference of the arcs into which the baſe is divided by the perpendicular arc AN. Q. E. D.

Cor. Hence, if each of the three ſides of any ſpherical triangle BAC be given, the angles are each of them found.

Let the baſe CB be 70°, the ſide AC 54°, and the ſide AB 32°, to find an angle ACB. The arcs CN, BN, into which the baſe is divided by the perpendicular arc AN, are each of them found as follows, by this propoſition. The half of the baſe is 35°, the half of the ſum of the two ſides is 43°, and the half of the difference of the ſides is 11°.

(1) By the logarithmic tables. Add 9.2886523, which is the logarithmic tan-

gent of 11°, to 9.9696559, which is the logarithmic tangent of 43°; and from the ſum 19.2583082 ſubtract 9.8452268, which is the logarithmic tangent of 35°; the remainder 9.4130814 is the logarithmic tangent of half the difference of the arcs CN, BN, 14° and between 30 and 31 minutes.

(2) By the logarithmic ſcales. Extend the compaſſes on the line of tangents from the tangent of 35° to the tangent of 43°; the ſame diſtance ſhall reach from the tangent of 11° to the tangent of 14° and 30 or 31 minutes.

(3) By the ſector. Make the lateral tangent of 43° the parallel tangent of 35°; then the parallel tangent of 11° ſhall be the lateral tangent of 14°, and 30 or 31 minutes.

To the half of the baſe CB, viz. 35°, add 14° 30′, which is half the difference of the arc CN, BN; and the ſum 49° 30′ ſhall be the greater arc CN; wherefore the other arc BN is 20° 30′. And becauſe AC, the hypothenuſe of the right-angled ſpherical triangle CAN, and a ſide CN, are each of them given, the angle ACN is found by caſe 7. of right-angled ſpherical triangles. In like manner, the angle ABN is found; then the angle at the vertex BAC is found,

either

either by finding the angles CAN, BAN, ſeparately, by caſe 1. of right-angled ſpherical triangles, and adding them into one ſum, or by cor. 1. 10. of this.

PROP. XVI. PROB. *Fig.* 6.

35. *To reduce any ſpherical triangle* ABC *to another ſpherical triangle* DEF, *ſo that each of the ſides of the triangle* DEF *ſhall be equal to the meaſure of each of the angles of the triangle* ABC, *(or of the angles adjacent to them, if any of them be obtuſe); and that the meaſure of each of the angles of the triangle* DEF *ſhall be equal to each of the ſides of the triangle* ABC, *or to the exceſs of a ſemicircle above any of the ſides of* ABC, *which is greater than a quadrant.*

Complete the circle ABKQ [25. 3.], and produce the arcs BC and AC, till they meet ABKQ again in S and R. From the pole A, at a quadrant's diſtance AG, deſcribe the circle GMQ, meeting AR in M; and from the pole B, at a quadrant's diſtance BK, deſcribe the circle KDP, meeting GMQ in D. Laſtly, from the pole C, at a quadrant's diſtance CH, deſcribe OHL, meeting AMR in L. Theſe are all great circles [16. 1. Theod.]. Becauſe ABKQ paſſes through the pole of GMQ, the pole of ABKQ [13. 1. Theod.] is in GMQ. For the ſame reaſon, the pole of ABKQ is in KDP; therefore it is the point D where they meet; and each of the arcs DG, DK, DQ, DP, is a quadrant: and becauſe ACL paſſes through the pole of each of the circles GMQ, LFO, the pole of ACL is the point F where they meet; therefore each of the arcs FM, FL, is a quadrant: therefore FM is equal to DG; and if the common part DM be taken from both, the remainder DF ſhall be equal to the remainder MG: but MG is [art. 3. of this] the meaſure of the angle BAC; therefore DF is equal to the meaſure of the angle BAC. And becauſe BCS paſſes through the pole of each of the circles KEP, LEO, the pole of BCS is the point E where they meet; therefore each of the arcs EN, EH, is a quadrant, and EH is equal to FL; therefore EF is equal to HL: but HL is the meaſure of the angle HCL, or of the vertical angle BCA; therefore EF is equal to the meaſure of the angle BCA. And becauſe EN and DK are each of them quadrants, ED is equal to NK: but NK is the meaſure of the angle NBK, which is adjacent to the obtuſe angle ABC.

Again,

Again, becauſe each of the arcs BN, CH, is a quadrant, BC is equal to NH; that is, to the meaſure of the angle DEF: And becauſe each of the arcs AM, CL, is a quadrant, AC is equal to ML; that is, to the meaſure of the angle MFL, which is adjacent to the obtuſe angle DFE: Likewiſe, becauſe each of the arcs AG, BK, is a quadrant, AB is equal to GK; that is, to the meaſure of the angle GDK, or of the vertical angle EDF: therefore the ſpherical triangle DEF is ſuch, that each of its ſides is equal to the meaſure of each of the angles of the ſpherical triangle ABC, or of the angles adjacent to ſuch of them as are obtuſe; and the meaſure of each of its angles, equal to each of the ſides of the triangle ABC, or to the exceſs of a ſemicircle above ſuch of them as are greater than quadrants. Which was to be done.

Cor. 1. Hence, if each of the angles ABC, BCA, BAC, of a ſpherical triangle ABC, fig. 6. be given, the ſides are each of them found: for, becauſe each of the angles of the triangle ABC is given, each of the ſides of the ſpherical triangle DEF is by this propoſition given; therefore [cor. 15. of this] each of the angles of the triangle

DEF is found, that is, each of the ſides of the ſpherical triangle ABC.

Cor. 2. Hence the tangent of half the angle BAC, fig. 5. at the vertex A of any ſpherical triangle ACB, or the tangent of half the adjacent angle GAB, if BAC be obtuſe, is to the tangent of half the ſum of the angles ABC and ACB, at the baſe BC, as the tangent of half the difference of the angles ABC, ACB, to the tangent of half the difference of the angles CAN, BAN, into which the angle BAC, or its adjacent GAB, is divided by the perpendicular arc AN.

36. Napier's two rules [art. 23. & 24. of this] may for the moſt part be uſefully applied in ſolving the caſes of oblique-angled ſpherical triangles. For, becauſe the rectangle contained by radius, and the ſine of the middle part, is equal to the rectangle contained by the coſines of the oppoſite extremes; and that the rectangle contained by radius and the ſine of the middle part is alſo equal to the rectangle contained by the tangents of the adjacent extremes; therefore the rectangle contained by the coſines of the oppoſite extremes is equal to the rectangle contained by the tangents of the

adjacent extremes: therefore [14. 6.] a ſide of the firſt rectangle is to a ſide of the ſecond, as the remaining ſide of the ſecond to the remaining ſide of the firſt.

Of the number of caſes in the ſolutions of oblique-angled ſpherical triangles.

37. Three of the ſix parts of a ſpherical triangle muſt be given in order to find a fourth. Two general caſes are deduced from prop. 10. viz. (1) When two ſides, and an angle oppoſite to one of theſe ſides, are each of them given, in order to find the angle oppoſite to the other ſide: Or, (2) When two angles, and a ſide oppoſite to one of theſe angles, are each of them given, in order to find the ſide oppoſite to the other angle. Theſe two caſes are ſolved likewiſe by Napier's firſt rule: for if the perpendicular arc AN, fig. 5. be the middle part, the complement of AB, the hypothenuſe of the right-angled ſpherical triangle ABN, and the complement of the angle ABN, ſhall be the oppoſite extremes. The ſame arc AN being the middle part, the complement of AC, the hypothenuſe of of the right-angled ſpherical triangle ACN,

and the complement of the angle ACN, ſhall be the oppoſite extremes; therefore [rule 1.] the rectangle contained by the ſine of AB, and the ſine of the angle ABN, is equal to the rectangle contained by the ſine of AC and the ſine of the angle ACN; and [14. 6.] a ſide of the firſt rectangle is to a ſide of the ſecond rectangle, as the remaining ſide of the ſecond to the remaining ſide of the firſt.

38. (1) If two ſides AC, AB, and an angle ACB, oppoſite to one of them, be each of them given, the baſe BC is found by means of prop. 11. For, becauſe the hypothenuſe AC, and an angle ACN of the right-angled ſpherical triangle ACN, are each of them given, the ſide CN is found by caſe 8. of right-angled ſpherical triangles; and then BN is found [cor. 1. 11. of this]: Or, if the angle ABC be found [art. 37. of this], the arcs CN, BN, are each of them found by rule 2. the complements of the angles ACB, ABC, being the middle parts.

(2) Likewiſe, if two ſides AC, CB, and the angle ACB, contained by theſe ſides, be each of them given, the other ſide AB is found: For the arc CN is found as in the preceding caſe; then the arc BN is given;

and [cor. 2. 11. of this] the coſine of CN is to the coſine of BN, as the coſine of AC to the coſine of AB.

39. (1) If two angles ACB, ABC, and a ſide AC, oppoſite to one of theſe angles, be each of them given, the remaining angle BAC is found by means of prop. 12. For becauſe the hypothenuſe CA, and the angle ACN, of the right-angled ſpherical triangle ACN, are each of them given, the angle CAN is found by caſe 10. of right-angled ſpherical triangles; and then the angle BAN is found [cor. 1. 12. of this]: Or, if the ſide AB be found [art. 37. of this], each of the angles CAN, BAN, is found by rule 2. the complements of AC and of AB being middle parts.

(2) Likewiſe, if two angles ACB, CAB, and the adjacent ſide AC, be each of them given, the other angle ABC is found: for the angle CAN is found as in the preceding caſe; then the angle BAN is given; and [cor. 2. 12. of this] the ſine of CAN is to the ſine of BAN, as the coſine of ACB to the coſine of ABC.

40. (1) If two angles ACB, ABC, and a ſide AC, oppoſite to one of the angles, be each of them given, the ſide CB, adjacent

to them, is found by means of prop. 15. For, becauſe the hypothenuſe AC, and the angle ACN, of the right-angled ſpherical triangle CAN, are each of them given, the ſide CN is found by caſe 8. of right-angled ſpherical triangles; then the arc BN is found [cor. 1. 13. of this]; becauſe the tangent of the angle ABN is to the tangent of the angle ACN, as the ſine of CN to the ſine of BN: Or if the ſide AB be found, each of the arcs CN, BN, is found by rule 2. the complements of the angles ACB, ABC, being the middle parts.

(2) Likewiſe, if two ſides AC, CB, and the angle ACB which theſe ſides contain, be each of them given, either of the other angles ABC is found: For the arc CN is found as in the preceding caſe; then the arc BN is given: and [cor. 2. 13. of this] the ſine of BN is to the ſine of CN, as the tangent of ACN to the tangent of ABN.

41. (1) If two ſides AC, AB, and an angle ABC oppoſite to one of theſe ſides, be each of them given, the angle BAC contained by them is found by means of prop. 14. For, becauſe the hypothenuſe AB, and the angle ABN, of the right-angled ſpherical

triangle ANB, are each of them given, the angle BAN is found by caſe 10. of right-angled ſpherical triangles; then the angle CAN is found [cor. 1. 14. of this]: Or if the angle ACB be found, the angles CAN, BAN, are each of them found by rule 2. the complements of AC and AB being the middle parts.

(2) Likewiſe, if two angles ABC, BAC, and the adjacent ſide AB, be each of them given, the ſide AC, oppoſite to ABC, either of the given angles, is found: For the angle BAN is found as in the preceding caſe; then the angle CAN is given: and [cor. 2. 14. of this] the coſine of the angle CAN is to the coſine of BAN, as the tangent of AB to the tangent of AC.

42. If the three ſides be each of them given, the angles are found by means of prop. 15.: and if the three angles be each of them given, the ſides are found by means of prop. 16. There are therefore twelve caſes of oblique-angled ſpherical triangles, which are ſet down in the following table.

Solutions of the caſes of oblique-angled ſpherical triangles. Plate 7. fig. 5.

	Given.	Sought.	Proportionals.
1.	Two ſides AC, AB, and the angle ABC, oppoſite to one of them.	The angle ACB, oppoſite to the other ſide.	The ſine of AC is to the ſine of AB, as the ſine of ABC to the ſine of ACB, [cor. 1. 10. of this]. This and the other caſes depending upon the ſame prop. are often ambiguous.
2.	Two angles ABC, ACB, and a ſide AC oppoſite to one of them.	The ſide AB, oppoſite to the other angle	The ſine of ABC is to the ſine of ACB, as the ſine of AC to the ſine of AB, [cor. 2. 10. of this.]
3.	Two ſides AC, AB, and an angle ACB oppoſite to one of them.	The baſe BC.	CN is found [caſe 8. right-angled ſpherical triangles]; then the coſine of AC is to the coſine of AB, as the coſine of CN to the coſine of BN, [cor. 1. 11. of this.]
4.	Two ſides AC, CB, and the angle ACB contained by theſe ſides.	The remaining ſide AB.	The arc CN is found as in the laſt caſe; then BN is given: and the coſine of CN is to the coſine of BN, as the coſine of AC to the coſine of AB, [cor. 2. 11. of this.]
5.	Two angles ACB, ABC, and a ſide AC oppoſite to one of the given angles.	The remaining angle BAC	The angle CAN is found [caſe 10. right-angled ſpherical triangles]; then the coſine of ACB is to the coſine of ABC, as the ſine of CAN to the ſine of BAN, [cor. 1. 12. of this.]
6.	Two angles ACB, CAB, and the adjacent ſide AC.	The remaining angle AB	The angle CAN is found as in the laſt caſe; then the angle BAN is given: and the ſine of CAN is to the ſine of BAN, as the coſine of ACB to the coſine of ABC, [cor. 2. 12. of this.]

	Given.	Sought.	Proportionals.
7.	Two angles ACB, ABC, and a ſide AC oppoſite to one of the given angles.	The ſide CB adjacent to the given angles.	CN is found [caſe 8. right-angled ſpherical triangles]; then the tangent of ABN is to the tangent of ACN, as the ſine of CN to the ſine of BN, [cor. 1. 13. of this.]
8.	Two ſides AC, CB, and the angle ACB which theſe ſides contain	Either of the other angles ABC.	CN is found as in the laſt caſe; then BN is given: and the ſine of BN is to the ſine of CN, as the tangent of ACB to the tangent of ABC, [cor. 2. 13. of this.]
9.	Two ſides AC, AB, and an angle ABC oppoſite to one of them.	The angle BAC which theſe ſides contain.	The angle BAN is found [caſe 10. right-angled ſpherical triangles]; then the tangent of AC is to the tangent of AB, as the coſine of BAN to the coſine of CAN, [cor. 1. 14. of this.]
10	Two angles ABC, BAC, and the adjacent ſide AB.	The ſide AC, oppoſite to ABC, either of the given angles.	The angle BAN is found as in the laſt caſe; then the angle CAN is given: and the coſine of CAN is to the coſine of BAN, as the tangent of AB to the tangent of AC, [cor. 2. 14. of this.]
11	Each of the three ſides AC, AB, and CB.	Each of the angles.	The tangent of the half of CB is to the tangent of half the ſum of AC and AB, as the tangent of half the difference of AC, AB, to the tangent of half the difference of CN, NB; wherefore the angles are each of them found, [cor. 15. of this.]
12	Each of the three angles ACB, ABC, and BAC.	Each of the ſides	The tangent of half the angle BAC at the vertex, or the tangent of half the adjacent angle GAB, is to the tangent of half the ſum of the angles ABC and ACB, as the tangent of half the difference of ABC, ACB, to the tangent of half the difference of CAN, BAN.

SECT. IV.

Examples in the Projection of the Sphere, and in Spherical Trigonometry.

EXAMPLE I. *Plate 7. fig. 7.*

1. *To deſcribe a map of the earth upon the plane of a given meridian* ABCD, *the eye being ſuppoſed to be in the pole of that meridian.*

Draw two diameters AC, BD, at right angles to one another; and let B, D, be the poles of the equator. The hour-circles, or meridians that divide the equator into twenty-four equal parts, are inclined to one another at an angle of fifteen degrees. From the centre E, therefore, make EF equal to the tangent of fifteen degrees; and from the centre F, with the diſtance FB or FD, deſcribe a circle; this circle [3. ſect. 2. 2.] ſhall be inclined to ABCD at an angle of fifteen degrees. In like manner, if EG be made equal to the tangent of thirty degrees, and if a circle be deſcribed from the centre G, with the diſtance GB or GD, it ſhall be inclined to ABCD at an angle of thirty degrees, and to the preceding at an angle of fifteen degrees. The reſt of the hour-circles, or any number of meridians, are deſcribed the ſame way, viz. by taking off from EC and EA, or from theſe lines produced, parts equal to the tangents of each meridian's inclination to ABCD.

The tropics PQ, NO, are deſcribed [5. ſect. 2. 2.], by taking from EB and ED ſtraight lines, each of them equal to the ſecant of the diſtance of the tropic from the pole, viz. ſixty-ſix degrees and an half: the points P, N, through which the tropics paſs, are found, by making the arcs AP, AN, each twenty-three degrees and an half [art. 11. ſect. 3. 1]; or, by making BP, DN, each ſixty-ſix degrees and an half: the circles deſcribed from the points found in EB, ED, as centres with the diſtances of theſe points from P and from N, ſhall be the tropics PQ, NO.

The polar circles HQ, ML, are deſcribed by taking off from EB and ED ſtraight lines, each equal to the ſecant of twenty-three degrees and an half; and the points H, and through which the polar circles paſs, are found, by making the arcs BH, DM, each twenty-three degrees and an half: the circles deſcribed from the points found in EB, ED, as centres with the diſtances of theſe points from H, M, ſhall be the tropics HK, ML. Any number of circles parallel to the equator may be deſcribed in the ſame manner.

If ABCD be the meridian which is in the plane of the ſolſtitial colure; the ecliptic PO is [art. 4. ſect. 2. 2.] projected into a ſtraight line. The equator AC, and the meridian BD, whoſe plane paſſes through the eye, are likewiſe in this example projected into ſtraight lines: and it is manifeſt, how places, whoſe longitudes and latitudes are known, are to be delineated in this projection.

It is manifeſt, from art. 9. and prop. 1. ſect. 2. 2. how a map is deſcribed upon the plane of the equator. In ſuch a map, the parallels of latitude are concentric with the primitive; and the meridians are ſtraight lines from the centre of the primitive to its circumference.

N. B. Theſe projections are founded upon the ſuppoſition, that the earth is a perfect ſphere. The diameter of the equator, however, is to the axis, as 230 to 229.

See *Newt. Prin. lib.* 3 *prop.* 19. *edit.* 1726. *Oeuv. de Mauper. tom.* 4. *p.* 335. *edit.* 1768. But this difference is ſo ſmall, that it would require a very large map to make it ſenſible.

EXAMPLE II. *Fig.* 8.

2. *To project the celeſtial ſphere upon the plane of the ſolſtitial colure* ADBE *for the horizon of any latitude, the eye being ſuppoſed at ſuch a diſtance, as that the rays of light may be conſidered as proceeding in parallel ſtraight lines.*

Let AB be the horizon, and make BD, AE, each of them [art. 11. or 41. ſect. 3. 1.] equal to the latitude, and join DE; D, E, ſhall repreſent the poles of the equator, and DE ſhall repreſent its axis, or the meridian whoſe plane paſſes through the eye; that is, the equinoctial colure. If the diameter FG be drawn at right angles to DE, it ſhall repreſent the equator. Make FQ, GS, each of twenty-three degrees and an half, and join QS, QS ſhall repreſent the ecliptic. The tropics and polar circles in this example are [art. 15. ſect. 2. 2.] projected into ſtraight lines. If therefore QP and SR be drawn parallel to FG, they ſhall repreſent the tropics. Make each of the arcs DN, DH, EK, EO, twenty-three degrees and an half, and join NH, KO; NH, KO, repreſent the polar circles: and if HK be joined, it ſhall repreſent the axis of the ecliptic. The meridians that are inclined to the plane of the projection, are in this example [8. ſect. 2. 2.] projected into ellipſes: the greater axis of each of theſe ellipſes is DE, and the leſſer axis of each of them is twice the coſine of the inclination to the primitive FDGE: if therefore CT and CU be each of them made equal to the coſine of fifteen degrees [art. 13. & 31. ſect. 3. 1], and if with the axes DE, TU, an ellipſis be deſcribed, this ellipſis ſhall repreſent the meridian which is inclined to the plane of the projection at an angle of fifteen degrees, that is, one hour in time. Any number of meridians may be deſcribed in the ſame manner. The ecliptic QS is divided into degrees by cutting off, towards the extremities Q, S, ſtraight lines equal to the verſed ſines of the number of degrees required [art. 15. ſect. 2. 2.]. Thus, if it be required to find the point in CQ, repreſenting the firſt of Gemini; becauſe Q repreſents the firſt of

Cancer,

Cancer, take off the primitive from Q towards N, and from Q towards A, two arcs, each of them thirty degrees, the ſtraight line joining the points ſo found in the primitive ſhall cut CQ in the firſt of Gemini: the ſame point is found by taking off from C towards Q, a ſtraight line equal to the coſine of thirty degrees [def. 9. ſect. 1. 1.]. In like manner, the repreſentation of any other point in the ecliptic is found. In this example, the firſt point of Aries coincides with the firſt point of Libra, and they are in the centre of the *primitive*; the firſt point of Taurus coincides with the firſt point of Virgo; the firſt point of Gemini coincides with the firſt point of Leo; and ſo on, except Cancer and Capricorn. The ellipſes repreſenting the meridians may be divided into parts repreſenting any number of degrees, by dividing the primitive into degrees, and drawing ſtraight lines from the points of diviſion in the primitive at right angles to DE; theſe ſtraight lines ſhall divide the ellipſes into parts repreſenting the parts of the primitive [cor. 8. ſect. 2. 2.].

EXAMPLE III. *Fig.* 9.

3. *To deſcribe a ſun-dial upon a given plane for a given latitude.*

The principles of dialling amount to no more than this: Any number of planes paſſing through the axis of a ſphere, and inclined to one another in a given angle, being given, to find the common ſection of each of theſe planes, with another plane given in poſition, which cuts each of them.

When this general problem is applied to dialling, the planes that are conſidered as paſſing through the axis of the ſphere are the planes of any given number of meridians dividing the equator into given parts; the plane given in poſition, cutting the plane of each of theſe meridians, and whoſe common ſection with each of them is ſought, is the plane upon which the dial is to be deſcribed; and theſe common ſections are the lines that ſhow the hours, or parts of hours, by means of a ſhadow cauſed by a line coinciding with the axis of the earth.

The angles at which the meridians are inclined to one another are determined by the time. Thus, if the number of meridians be twelve, the time correſponding to each arc of the equator, which is the meaſure of the inclination of any one of them to the next, is one hour, and that arc of the equator is fifteen degrees; the arc of the equator correſponding to two hours is thirty degrees; and the arc correſponding to half an hour is ſeven degrees and an half, and ſo on. The poſition of the plane cutting the planes of the meridians, viz. the plane upon which the dial is to be deſcribed, is determined by the latitude of the place for which the dial is to be made, and by the inclination the maker pleaſes to give this plane to the eaſt, weſt, north, or ſouth. Hence the conſtruction of a dial is reduced to a caſe of right-angled ſpherical triangles. The following is intended as an example of an horizontal ſun-dial; and the hour-lines are found upon it, firſt by calculation, and from thence is deduced a method of finding them by geometrical conſtruction.

Draw F XII a diameter of the circle VI XII VI, in F XII take any point A, and through A draw VI VI at right angles to F XII, and from A let a ſtraight line be raiſed above the plane, containing an angle with A XII equal to the latitude of the place, this ſtraight line is to repreſent the axis of the earth. Then, if F XII repreſent the north and ſouth line, viz. the direction of the ſhadow proceeding from A at twelve o'clock, VI VI ſhall repreſent the eaſt and weſt line, viz. that in which the ſhadow is at ſix o'clock in the morning, and at ſix o'clock in the evening. In order to find AI, viz. the common ſection of the meridian paſſing through the point of the equator that correſponds to one o'clock in the afternoon, with the plane of the horizon or dial, there are given one ſide of a right-angled ſpherical triangle, and the adjacent ſpherical angle, in order to find the ſide oppoſite to this angle; and the ſide that is required is the meaſure of the angle XII A I. The ſide that is given is the elevation of the pole above the horizon, which is equal to the latitude of the place: the angle that is given is meaſured by the number of degrees of the equator intercepted between the meridian correſponding to twelve o'clock, and the meridian correſponding to one o'clock in the afternoon, viz. fifteen degrees: and this angle is vertical,

tical, and therefore equal to the angle contained by theſe meridians on the other ſide of the pole; for the continuation of the meridian paſſing through the point of the equator that correſponds to one o'clock in the afternoon beyond the pole, is the hypothenuſe of the ſpherical triangle in queſtion. This example, therefore, is the fifth caſe of right-angled ſpherical triangles.

Let the latitude of the place be 56°, to find the angle XII A I. Add 9.4280525, which is the logarithmic tangent of 15° correſponding to one hour, to 9.9185742, which is the logarithmic ſine of 56°, viz. the latitude of the place; and from the ſum 19.3494267 ſubtract 10.0000000, which is the logarithm of radius; the remainder 9.3494267 is the logarithmic tangent of the angle XII A I 12° 36′: if therefore, at the point A, in the ſtraight line A XII, the angle XII A I be made [art. 11. or 41. ſect. 3. 1.] of 12° 36′, the ſtraight line AI ſhall be the direction of the ſhadow proceeding from A at one o'clock in the afternoon, in latitude 56°. Likewiſe, the number of degrees of the angle XII A II is found; becauſe its tangent is the fourth proportional to the three terms, radius, the tangent of

30°,

30°, (correſponding to two hours), and the ſine of 56°. In like manner, the direction of the ſhadow at each hour or part of an hour, till ſix o'clock in the afternoon, is found. The angle contained by each hour-line, half-hour line, and quarter of an hour-line, &c. in one quadrant being thus found, they are transferred to the other quadrant; and then the lines in which the ſhadow is at each hour, and part of an hour, from ſix o'clock in the morning till twelve o'clock, ſhall be determined; for the angle contained by the ſhadow, and by A XII at eleven o'clock in the forenoon, is equal to that contained by the ſhadow, and by A XII at one o'clock in the afternoon. The lines in which the ſhadow is at any hour, or part of an hour, after ſix in the evening, and before ſix in the morning, ſo long or ſo early as the ſun is above the horizon in the given latitude, are found, by making at the point A, in the ſtraight line A VI, on the other ſide, an angle equal to the angle VI A V; the line containing this angle with A VI ſhall be the poſition of the ſhadow at ſeven o'clock in the evening. In like manner, if an angle be made at the point A, on the ſame ſide of A VI, with the laſt angle, equal to the angle VI A IV, it ſhall give the poſition of the ſhadow at eight o'clock at night, and ſo on; and if angles equal to theſe be made at the point A, in A VI, on the other ſide of AF, the poſition of the ſhadow at five and four o'clock in the morning ſhall be determined.

From the proportions by which the poſition of the ſhadow of the ſtile, at any time of the day, is found by ſpherical trigonometry, the following geometrical conſtruction of an horizontal ſun-dial is deduced, which, ſo far as I know, is the eaſieſt in practice.

Make the angle XII AB equal to the latitude of the place, or to the elevation of the pole above the horizon; in A XII take any point D; and through D draw DG at right angles to A XII; from D draw DC at right angles to AB, meeting it in C; make DE upon D XII equal to DC; and from the centre E, with the diſtance ED, deſcribe a quadrant of a circle; divide this quadrant [15. 4. & 30. 3.] into ſix equal parts; and draw a ſtraight line from the centre E through each point of diviſion in the quadrant, meeting DG; then from A draw a ſtraight line through each of the points ſo found in DG, and produce them to I, II, III, &c.; then if the ſtile be raiſed from A above the plane of the dial, containing with AE an angle equal to the latitude of the place, and if AE be placed north and ſouth, the ſtraight lines A I, A II, A III, &c. ſhall be the poſition of the ſhadow at one, two, three, &c. o'clock in the afternoon; and if the parts into which DG is divided be transferred to DB, they ſhall give the poſition of the ſhadow in the forenoon; and if the quadrant deſcribed from the centre E, with the diſtance ED, be divided into twelve equal parts, and if ſtraight lines be drawn from the centre E through each point of theſe diviſions, meeting DG, the points ſo found in DG ſhall give the poſition of the ſhadow each half hour, and ſo on. It is manifeſt [1. ſect. 4. 1.], that when AD, the hypothenuſe of the right-angled plane triangle ADC, is made radius, DC is the ſine of the angle DAC: but the angle DAC, by conſtruction, is equal to the latitude of the place, or to the elevation of the pole above the horizon of the place; ED likewiſe, by conſtruction, is equal to DC; and DL, which [def. 3. ſect. 1. 1.] is the tangent of the angle DEL when ED is made radius, is likewiſe the tangent of DAL when AD is made radius.

Hitherto the ſtile of the dial is conſidered as having no thickneſs; and indeed a very thin ſtile is requiſite, that the direction of the ſhadow may be accurately obtained, eſpecially about twelve o'clock. But becauſe a very thin ſtile would be liable to accidents, a thick ſtile is commonly uſed; ſo that the common ſection of each of the planes, which are the eaſt and weſt ſides of it, with the other plane, is conſidered as a ſtile. The one of theſe common ſections, or edges of the ſtile, which is towards the eaſt, ſhows the time of the day from twelve o'clock till ſix in the afternoon: but the edge that is towards the weſt ſhows it after ſix in the afternoon; the eaſtern edge again, ſhows the time of the day before ſix o'clock in the morning; and the weſtern edge ſhows it from ſix in the morning till twelve o'clock, as is manifeſt; and therefore the hour-lines before ſix in the morning are drawn from the eaſtern centre; and the hour-lines after ſix in the evening are drawn from the weſtern centre. The common ſections of the eaſtern and weſtern ſides of the ſtile, with the plane of the dial, are drawn, making a blank ſpace at XII equal to the thickneſs of the dial. In places that have much north latitude, the point A where the ſtile is fixed, is taken conſiderably to the ſouth of the centre of the dial: in ſouth latitude, it is probably taken to the north.

The common method of finding the north and ſouth, or meridian line for a dial, is to deſcribe a circle upon a horizontal plane; and from the centre of this circle, to erect a ſtile at right angles to the plane, and to obſerve the two points in the circumference of the circle in which the extremity of the ſhadow is before and after noon, at a time when the ſun does not ſenſibly alter his declination in a few hours; the ſtraight line drawn from the centre of the circle biſecting the arc, intercepted between theſe two points, is the meridian line. The ſummer ſolſtice is the fitteſt time for it. Tho' this method may ſerve for common uſe, it is not to be depended upon alone for the more important aſtronomical obſervations.

In the following examples, no reference is made to any figure; becauſe the problems comprehended in them are intended to be ſolved upon a globe, in order to exerciſe the learner upon particular caſes of ſpherical triangles, and to ſhew him their uſes.

EXAMPLE IV.

4. *The ſun's place in the ecliptic, and the inclination of the ecliptic to the equator, being each of them given, to find the ſun's right aſcenſion; that is, the point of the equator that comes to the meridian with the ſun.*

Let the ſun's place be the twenty-fifth degree of Taurus, and the angle contained by the ecliptic and equator, or, as it is called, the declination of the firſt point of Cancer and of Capricorn 23° 29′, to find the ſun's right aſcenſion. Becauſe the ſun's place is the 25° of Taurus, his diſtance from the firſt point of Aries, that is, his longitude, is 55°. The example therefore is the eighth caſe of right-angled ſpherical triangles: for the ſun's longitude is the hypothenuſe, the inclination of the ecliptic and equator is the given angle, and the right aſcenſion which is ſought is the ſide adjacent to the given angle. When the right aſcenſion is found, it is reduced to time by reckoning an hour for each fifteen degrees, and a minute for each fifteen minutes, &c.

5. The ſame things being given, to find the ſun's declination; that is, the number of degrees upon the meridian that paſſes through the ſun, intercepted between the ſun and the equator. This is the ſecond caſe of right-angled ſpherical triangles.

6. The ſame things being given, to find the angle contained by the meridian paſſing through the ſun, and by the ecliptic. This is the tenth caſe of right-angled ſpherical triangles.

EXAMPLE V.

7. *The latitude of the place being given, and likewiſe the right aſcenſion and declination of the ſun being each of them given, to find the ſun's amplitude; that is, the number of degrees upon the horizon he riſes or ſets to the north or ſouth of eaſt and weſt.*

Becauſe the latitude of the place is given, the angle contained by the plane of the equator, and that of the horizon, is given, being the complement of the latitude, or of the elevation of the pole. Likewiſe the declination of the ſun is the ſide oppoſite to the given angle, and the ſpherical triangle is right-angled; therefore the hypothenuſe, which is the amplitude required, is found. This is the third caſe of right-angled ſpherical triangles.

8. The ſame things being given, to find the ſun's aſcenſional difference; that is, the difference between the number of degrees of the equator, which comes to the meridian with the ſun, and that which is at the horizon with the ſun. In the laſt right-angled ſpherical triangle, there are given a ſide, and the oppoſite angle; therefore the other ſide is found. This is the ſixth caſe of right-angled ſpherical triangles. And becauſe the right aſcenſion is given, and that the aſcenſional difference is thus found, the oblique aſcenſion, that is, the point of the equator which is at the horizon with the ſun in an oblique ſphere, is found, by ſubtracting or adding the aſcenſional difference.

EXAMPLE VI.

9. *The longitude and latitude of any ſtar without the ecliptic, and the inclination of the ecliptic to the equator, being each of them given, to find the ſtar's declination.*

Becauſe the longitude of the ſtar is given, the angle contained by its circle of longitude, and by the ſolſtitial colure, is given: and becauſe the latitude of the ſtar is given, the complement of its latitude, that is, the arc of the circle of longitude intercepted between the ſtar and the pole of the ecliptic, is given; and the diſtance between the poles of the ecliptic and equator is given: there are therefore two ſides of an oblique-angled ſpherical triangle; and the angle which theſe ſides contain, given in order to find the third ſide, and the third ſide, is an arc of a great circle intercepted between the ſtar and the pole of the equator, which is the complement of the ſtar's declination. This is the fourth caſe of oblique-angled ſpherical triangles.

10. The ſame things being given, to find the ſtar's right aſcenſion. In the laſt ſpherical triangle, the angle contained by the ſolſtitial

colure, and by the arc paſſing through the pole of the equator, and through the ſtar, is found by caſe 8. of oblique-angled ſpherical triangles; therefore the meaſure of this angle upon the equator is found. And becauſe the point of the equator, through which the ſolſtitial colure paſſes, is given, the point of the equator, through which the meridian of the ſtar paſſes, is likewiſe given; and this laſt point in the equator gives the right aſcenſion of the ſtar.

EXAMPLE VII.

11. *The declination and right aſcenſion of a ſtar being each of them given, and the declination of another ſtar being given, as alſo the diſtance between the two ſtars, to find the right aſcenſion of the latter ſtar.*

Becauſe the declination of each ſtar is given, the complement of the declination of each of them is given; that is, the diſtance of each ſtar from the pole of the equator; and becauſe the diſtance of the two ſtars from one another is likewiſe given, the three ſides of the ſpherical triangles in the angular points, of which the two ſtars and the pole of the equator are each of them given; therefore, by caſe 11. of oblique-angled ſpherical triangles, the ſpherical angle at the pole of the equator, contained by the complements of the declination of the two ſtars, is found: and this angle is meaſured upon the equator by the difference of right aſcenſion of the two ſtars; therefore, becauſe the right aſcenſion of one of the ſtars is given, the right aſcenſion of the other is found.

EXAMPLE VIII.

12. *The latitude and longitude of two ſtars being given, and the diſtance of another ſtar from each of them being likewiſe given, to find the latitude and longitude of this other ſtar.* See Greg. Aſtr. book 2. prop. 28.

Becauſe the latitude of each of the two ſtars is given, the complement of the latitude of each of them is given; and becauſe the longitude of each of them is given, their difference of longitude is given: but their difference of longitude is the meaſure of the ſpherical angle at the pole of the ecliptic which is contained by the circles of longitude of the two ſtars; therefore this angle is alſo given, and by the fourth caſe of oblique-angled ſpherical triangles, the diſtance of the two ſtars is found; likewiſe the ſpherical angle contained by the arc which is their diſtance, and by the arc which is the complement of the latitude of one of them, is found by the firſt caſe of oblique-angled ſpherical triangles. And becauſe the diſtance of the two ſtars whoſe longitudes and latitudes are given has been found, and that the diſtance of the third ſtar from each of them is given, the three ſides of the ſpherical triangle, in whoſe angular points theſe three ſtars are, are each of them given; therefore the angle contained by the arc which is the diſtance of the two firſt ſtars, and by the arc which is the diſtance of one of theſe from the third ſtar, is found by the eleventh caſe of oblique-angled ſpherical triangles; and if this angle be taken from the angle contained by the arc which is the diſtance of the two firſt ſtars, and by the arc which is the complement of the latitude of one of them, (which angle was already found), the remaining angle contained by the arc which is the diſtance of the ſtar, whoſe longitude and latitude is ſought from the one of the firſt two, and by the arc which is the complement of this ſtar's latitude, is given: and the ſides containing this laſt angle are each of them given; therefore the ſide oppoſite to the given angle is found by the fourth caſe of oblique-angled ſpherical triangles: and this ſide is the complement of the latitude of the ſtar whoſe latitude is ſought; therefore its latitude is found. And becauſe the diſtance of the ſtar whoſe latitude and longitude was ſought from the pole of the ecliptic has been found, and that its diſtance from the one of the firſt ſtars is given, and likewiſe that the angle adjacent to this laſt ſide has been found, the angle adjacent to the other ſide, viz. the angle at pole of the ecliptic, is found by the ſecond caſe of oblique-angled ſpherical triangles; and the meaſure of this angle upon the ecliptic is the difference of longitude of the ſtar, whoſe latitude and longitude was ſought from that of the one of thoſe ſtars whoſe latitudes and longitudes were given; therefore the latitude and longitude of the third ſtar is found.

FINIS.

Directions to the BINDER.

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